Game Theory, Day 1

Part I: Syllabus and Introductions

Part II: Chapter 1 of Osborne

- Discuss difference between game theory and price theory (no interactions between individuals in price theory)
- Do example of Centipede Game (bring quarters)
  - Induction (use example of first n integers)
  - Draw game tree
  - Describe backward induction
  - Discuss other solution concepts, explain relationship between Nash equilibrium and backward induction
- Do example of Prisoners' Dilemma
- Components of games
  - Players
  - Actions
  - Preferences
- Theory of rational choice
  - Preferences and utility functions
  - These are ordinal utility functions
  - More than one utility function representation of preferences
  - Criticisms of rational choice: "ex-hypothesis"
  - But we keep it because there's no alternative ("No general theory currently challenges the supremacy of rational choice theory.") Read Kuhn's *Structure of Scientific Revolutions* if you think this is weird.

Game Theory, Day 2

Part I: Calculus Review

See notes elsewhere (on health economics page).

Part II: Introduction to Strategic Games, Sections 2.1–2.5

- Formal definition of strategic game
  - Players
  - For each player, a set of possible actions. An **action profile** $a = (a_1, a_2, \ldots, a_n)$ contains one item from each player's set of actions. All together, the set of action profiles indicate all of the possible outcomes of the game.
  - For each player, preferences over the set of action profiles
- Note that there is no time element, but actions can be conditional
- Exercise with class: Determine these for Coke/Pepsi game
  - Note that preferences can be expressed with many different utility functions
- Note that Coke/Pepsi is one application of a Prisoners' Dilemma game
  - A reasonable definition: a symmetric game in which dominant strategies lead the players to an outcome that is not Pareto efficient
  - Other examples: arms race (build/don't build), common property (graze a little/graze a lot' see Tragedy of the Commons)
- Games that are not Prisoners' Dilemma games
  - BoS. Applications include competing preferences within a political party ("We must all hang together, or assuredly we shall all hang separately"– Benjamin Franklin at the signing of the Declaration of Independence); merging firms with different computer systems.
  - Matching pennies. A purely conflictual game, aka "zero sum" or "strictly competitive"
  - Stag hunt. An alternative model of the "security dilemma"

Nash equilibrium, Section 2.6

- What is a "good" solution concept for a strategic game?
- Nash equilibrium: the idea
  - Each player's action is optimal given her beliefs about the other players' actions (theory of rational choice)

  - Each player's beliefs about other players' actions are correct!
- How are beliefs formed and why should they be correct?
  - Good questions!
  - Beliefs are based on experience against "similar" players
- Nash equilibrium: more formally
  - A Nash equilibrium is an action profile $a^* = (a_1^*, a_2^*, \ldots, a_n^*)$ with the property that no player i can do better by choosing an action different from a_i^*, given that every other player j adheres to a_j^*.
  - In English, take 1: The players' actions are **mutual best responses**.
  - In English, take 2: The players' actions are **second-guess-proof**.
  - In English, take 3: No player can gain by deviating alone.
  - In biology: The introduction of a single mutation into a population will not dominate.
- Nash equilibrium: most formally
  - Let $a = (a_1, a_2, \ldots, a_n)$ be an action profile. Now let (a_i', a_{-i}) be the action profile where player i "deviates" by playing action a_i' while all other players adhere to a.
  - Formally, then, if $u_i(a)$ is the utility that player i gets from action profile a, the action profile a^* is a Nash equilibrium if $u_i(a^*) \geq u_i(a_i, a_{-i}^*)$ for all players i and all actions a_i.

Game Theory, Day 3

Braess's traffic paradox

Have class work on problem (on website)

Finish Nash equilibrium, Section 2.6

- Nash equilibrium: most formally (finish from above)
  - Let $a = (a_1, a_2, \ldots, a_n)$ be an action profile. Now let (a'_i, a_{-i}) be the action profile where player i "deviates" by playing action a'_i while all other players adhere to a.
  - Formally, then, if $u_i(a)$ is the utility that player i gets from action profile a, the action profile a^* is a Nash equilibrium if $u_i(a^*) \geq u_i(a_i, a^*_{-i})$ for all players i and all actions a_i.
- Finding Nash equilibrium with cross-outs and underlinings (hand-out on website)

Extensions of Nash equilibrium, Section 2.7

- Strict Nash equilibrium
  - A **strict Nash equilibrium** has $u_i(a^*) > u_i(a_i, a^*_{-i})$ for all players i and all actions a_i. (Strict inequality v. weak inequality.)

Strict dominance and weak dominance, Section 2.9

- Definition: We say that Player i's action a''_i **strictly dominates** action a'_i (and that a'_i is **strictly dominated by** a''_i) if $u_i(a''_i, a_{-i}) > u_i(a'_i, a_{-i})$ for all a_{-i}.
- Definition: We say that Player i's action a''_i **weakly dominates** action a'_i (and that a'_i is **weakly dominated by** a''_i) if $u_i(a''_i, a_{-i}) \geq u_i(a'_i, a_{-i})$ for all a_{-i}.
- A strictly dominated strategy is not used in any Nash equilibrium.
- Weakly dominated strategies *can* be used in Nash equilibrium.

Game Theory, Day 4

Cournot: Collusion, Duopoly, Oligopoly

Consider a good with demand curve $q(p) : q = \alpha - p$ for $p \geq 0$, i.e., with **inverse demand curve** $p(q) : p = \alpha - q$ for $q \leq \alpha$. (We have $p = 0$ for $q \geq \alpha$.) Imagine that production involves no fixed costs and constant marginal costs of $c \ll \alpha$.

Perfect competition

Under perfect competition, the supply curve is perfectly elastic at $p = c$ and the market price will always be c. Firms make zero profit.

Monopoly/Collusion

A monopolist chooses p and q to maximize

$$\pi = pq - cq = (p - c)q$$

subject to

$$p = \alpha - q.$$

Solve by substituting for p, so that you choose q to maximize

$$\pi = (\alpha - q - c)q = \alpha q - q^2 - cq.$$

At a maximum either $q = 0$ or $q = \infty$ or we have an interior maximum. We can easily dismiss the first two possibilities—in both cases the firm makes zero profit—and focus on the interior maximum, at which point the derivative of the profit function must be zero. Taking a derivative and setting it equal to zero we get

$$\alpha - 2q - c = 0, \text{ i.e., } q^* = \frac{1}{2}(\alpha - c).$$

This produces a price of

$$p^* = \frac{1}{2}(\alpha + c)$$

and profits of

$$\pi^* = \frac{1}{4}(\alpha - c)^2.$$

You can also think of this as the collusive outcome.

Cournot duopoly

Each firm chooses q_i, profits are $\pi_i = (p - c)q_i$ where $p = \alpha - q_1 - q_2$. To find the Nash equilibrium, note that firm 1 chooses $q_1 \geq 0$ to maximize

$$\pi_1 = (p - c)q_1 = (\alpha - q_1 - q_2 - c)q_1 = \alpha q_1 - q_1^2 - q_1 q_2 - cq_1.$$

At a maximum, either $q_1 = 0$ or $q_1 = \infty$ or (for an interior maximum) $\frac{d\pi}{dq_1} = 0$.

Maybe $q_1 = 0$ If there is a Nash equilibrium with $q_1 = 0$ then firm 2 must be producing the monopoly output level as determined above. But in this case the market price is $p^* = \frac{1}{2}(\alpha + c) \gg c$, so firm 1 is not acting optimally by producing zero output. So this cannot be part of a Nash equilibrium.

Maybe $q_1 = \infty$ If firm 1 produces an infinite amount of output then the market price will be zero, so firm 1 loses money; this cannot be part of a Nash equilibrium.

Interior solution Setting this partial derivative equal to zero gives us

$$\alpha - 2q_1 - q_2 - c = 0,$$

which solves to

$$q_1 = \frac{1}{2}(\alpha - q_2 - c).$$

This is a **best response function**: given q_2, it tells us what firm 1 should choose for q_1 in order to maximize profits.

Of course, Firm 2 chooses similarly. Its best response function is symmetric:

$$q_2 = \frac{1}{2}(\alpha - q_1 - c).$$

In a Nash equilibrium the two firms' choices will be mutual best responses, so we must have

$$q_1 = \frac{1}{2}(\alpha - q_2 - c) \text{ and } q_2 = \frac{1}{2}(\alpha - q_1 - c).$$

Solving these equations simultaneously (it helps to notice the symmetry, so we'll end up with $q_1 = q_2$) we get

$$2q_1 = \alpha - q_1 - c, \text{ i.e., } q_1 = q_2 = \frac{1}{3}(\alpha - c).$$

Total output is

$$q_T = \frac{2}{3}(\alpha - c),$$

so the market price will be

$$p = \alpha - q_T = \frac{1}{3}(\alpha + 2c).$$

Total profit (for both firms together) is

$$(p - c)q_T = \left[\frac{1}{3}(\alpha + 2c) - c\right] \cdot \frac{2}{3}(\alpha - c) = \frac{2}{9}(\alpha - c)^2,$$

which is less than under the monopoly situation.

- Why does it make sense for total profits to be less than under the monopoly situation?

Cournot oligopoly

Each firm chooses q_i, profits are $\pi_i = (p-c)q_i$ where $p = \alpha - (q_1 + \ldots + q_n)$. To find the Nash equilibrium, note that firm 1 chooses $q_1 \geq 0$ to maximize

$$\pi_1 = (p-c)q_1 = (\alpha - q_1 - \ldots - q_n - c)q_1 = \alpha q_1 - q_1^2 - q_1(q_2 + \ldots + q_n) - cq_1.$$

At a maximum, either $q_1 = 0$ or $q_1 = \infty$ or (for an interior maximum) $\frac{d\pi}{dq_1} = 0$. For reasons discussed previously we focus on the interior solution. Setting this partial derivative equal to zero gives us

$$\alpha - 2q_1 - (q_2 + \ldots + q_n) - c = 0,$$

which solves to

$$q_1 = \frac{1}{2}[\alpha - (q_2 + \ldots + q_n) - c].$$

This is a **best response function**: given $q_2, \ldots, q_n$, it tells us what firm 1 should choose for q_1 in order to maximize profits.

Of course, the other firms choose similarly. From symmetry it follows that $q_i^* = q_j^*$ in a Nash equilibrium, so we must have

$$q_i = \frac{1}{2}[\alpha - (n-1)q_i - c] \text{ , i.e., } q_i = \frac{1}{n+1}(\alpha - c).$$

Total output is

$$q_T = \frac{n}{n+1}(\alpha - c),$$

so the market price will be

$$p = \alpha - q_T = \frac{1}{n+1}(\alpha) + \frac{n}{n+1}(c).$$

Total profit (for all firms together) is

$$(p-c)q_T = \left[\frac{1}{n+1}(\alpha) + \frac{n}{n+1}(c) - c\right] \cdot \frac{n}{n+1}(\alpha - c) = \frac{n}{(n+1)^2}(\alpha - c)^2.$$

- The punch line: As the number of firms increase, p approaches c, profits approach zero, and game theory approaches price theory.

Game Theory, Day 5

Homework

Go over problems.

Class example?

Do first problem from PS 2.

Game Theory, Day 6

Bertrand

In the Bertrand model, firms compete on the basis of price rather than quantity. Firms that do not have the lowest price are shut out of the market; the firm or firms that do have the lowest price must meet all of the customer demand at that price, which is given by $q = \alpha - p$. (If there are multiple firms with the lowest price, some division rule must be established. We begin by assuming that the firms with the lowest price split the market.) Firms have constant unit costs of c.

Nash equilibrium

Note that there are no Nash equilibrium(s) with any price less than c. With a duopoly, there are also no Nash equilibrium with any price more than c. But (c, c) is a Nash equilibrium!

An indirect approach via best response functions

Firm 1's best response function looks like this:

- If $p_2 < c$, any $p_1 > p_2$ is a best response.
- If $p_2 = c$, any $p_1 \geq c$ is a best response.
- If $p_2 > c$, Firm 1 wants to choose a price just below p_2 in order to grab the market at the highest possible price. This is a problem (because we have continuous variables) unless p_2 is very high, in which case Firm 1 would choose the monopoly price p^m. So we actually have two cases here:
  - If $c < p_2 \leq p^m$, Firm 1 has no best response because it wants to choose p_1 just below p_2.
  - If $p_2 > p^m$, Firm 1's best response is to choose $p_1 = p^m$.

Comparing Cournot and Bertrand

In terms of the structure of the models, the difference between the two models is that Cournot competes on quantity and Bertrand competes on price. In terms of the predictions of the models, the difference is that Cournot makes a gradual transition between monopoly and perfect competition while Bertrand plunges from one to the other. The Bertrand firms are much more "on edge" because the market split is incredibly sensitive to small changes in price.

Finish class example

Do first problem from PS 2.

Game Theory, Day 7

Hotelling

What if competition is on the basis of something other than price or quantity? Consider two firms that compete on the basis of product characteristics, or two politicians who compete on the basis of a one-dimensional left-right dichotomy.

Popsicle trucks on the beach Assume a continuum of bathers who are spread out evenly and choose the closest truck (or choose randomly among the closest trucks if there are multiple such trucks).

- The only Nash equilibrium with two popsicle trucks is for them both to be in the middle.
- Show that there is no NE with three trucks because (i) if all three are in the same spot, one truck can deviate and get at least half the market; and (2) if all three are not in the same spot, one of them is an extremist and can gain by becoming less extreme.

A simple voting model

- Voters' preferences as in Figure 71.1.
- The **median voter** has a position m such that half the voters prefer $x \leq m$ and half the voters prefer $x \geq m$. (We assume there is exactly one such median voter position m.)
- Do example with two candidates and three voters, as in Figure 72.1.
- Show best response function, as in Figure 73.1.
- Show that both candidates gravitate to the median voter. Hotelling (1929, p. 54) writes that this outcome is "strikingly exemplified...The competition for votes between the Republican and Democratic parties [in the United States] does not lead to a clear drawing of issues, an adoption of two strongly contrasted positions between which the voter may choose. Instead, each party strives to make its platform as much like the other's as possible."
- Do or assign exercises 73.1 and 74.2. (I emailed Osborne with some questions about 74.1, so lay off that one for now.)

War of Attrition

"Waiting game" battle for item with value $v_i > 0$ for whoever wins it outright, and $\frac{1}{2}v_i$ if it's split equally. Cost of waiting until time t is t. Payoff functions as shown on p. 77.

Is there a NE where both players tie? No, because either player can gain by increasing their concession time by one millisecond.

Is there a NE in which one player wins and the other bids $t_i > 0$? No, because the losing player can gain by deviating to $t_i = 0$.

Is there a NE in which one player wins and the other bids $t_i = 0$? Yes, and in fact it can be *either* player. Any action profile with $t_1 = 0$ and $t_2 \geq v_1$ is a NE. So is any action profile with $t_2 = 0$ and $t_1 \geq v_2$.

- Do or assign exercises 80.1 and 80.2.

Game Theory, Day 8: Auctions

First consider an ascending price auction. Use the "having to go pee" story to show that strategies consist of maximal bids, i.e., of "stop" prices. This allows us to relate ascending price auctions to second-price sealed bid auctions!

Second-price sealed bid auctions

In a second-price sealed bid auction, bidders have values of v_i; we assume these are all different and that the bidders are numbered in such a way that

$$v_1 > v_2 > \ldots > v_n > 0.$$

Each bidder makes a bid of b_i. The winning bidder is the one with the highest bid; ties are broken in such a way that the bidder with the highest value wins, i.e., a tie between bidders n and m is won by bidder n iff $v_n > v_m$, i.e., iff $n < m$. If player i wins, her payoff is $v_i - b_j$, where b_j is the highest bid submitted by the other bidders. Players who don't win get a payoff of zero.

Show equivalence using eBay Proxy bidding overhead.

Identify weakly dominant strategies Show that bidding your true value is a weakly dominant strategy, meaning that you cannot do any better than bidding your true value. (If you lose, the highest bid is $\geq$ your true value; if you change your bid to an amount $<$ this highest bid, you still lose; if you change your bid to an amount $>$ this highest bid, you will end up bidding—and paying—an amount equal to or more than your true value, which does not improve your payoff; if you change your bid to an amount $=$ this highest bid, you either lose the auction or win and pay an amount $\geq$ your true value. If you win, the second-highest bid is $\leq$ your true value; if you change your bid to an amount $>$ this second-highest bid, you still win and pay that second-highest bid; if you change your bid to an amount $<$ this second-highest bid, you lose the auction, which does not improve your payoff; if you change your bid to an amount $=$ this second-highest bid, you either lose the auction or win and pay the same amount as before.

Identify weakly dominant NE If bidders bid their true values, the outcome is $(v_1, v_2, \ldots, v_n)$ and bidder 1 wins and pays a price of v_2. This is a Nash equilibrium because nobody can gain by deviating. Bidding your true value is a (weakly) dominant strategy for every player.

Identify other NEs Other equilibriums include $(v_1, 0, \ldots, 0)$, in which case player 1 wins and pays nothing, and $(v_2, v_1, 0, \ldots, 0)$, in which case player 2 wins and pays v_2.

Make the case for the weakly dominant NE The other equilibriums are unlikely because players are not playing their weakly dominant strategies.

Homework Assign 84.1 and (maybe?) 86.1. (No, but do Groves-Clark mechanism at some point.)

First-price sealed bid auctions

In a first-price sealed bid auction, bidders have values of v_i; we assume these are all different and that the bidders are numbered in such a way that

$$v_1 > v_2 > \ldots > v_n > 0.$$

Each bidder makes a bid of b_i. The winning bidder is the one with the highest bid; ties are broken in such a way that the bidder with the highest value wins, i.e., a tie between bidders n and m is won by bidder n iff $v_n > v_m$, i.e., iff $n < m$. If player i wins, her payoff is $v_i - b_i$. Players who don't win get a payoff of zero.

Equivalence with descending price auctions Strategies consist of "stop" bid. "Having to go pee."

In any NE, bidder 1 wins and pays $\leq v_1$ and $\geq v_2$ If someone else wins, either they bid $\geq v_1$, in which case they can deviate to a bid of zero, or they bid $< v_1$, in which case bidder 1 can deviate to a slightly higher bid.

In any NE, someone else must bid the same amount as bidder 1 Otherwise bidder 1 can deviate to ϵ less.

Description of all NE A set of bids is a NE iff both of the above conditions are met.

How to decide between these NE? Let's see if any strategies are dominant or dominated.

Bidding $\geq$ your true value is a weakly *dominated* action. It's weakly dominated by any bid < your true value because bidding your true value always gets you a payoff of zero.

Bidding $0 < b_i < v_i$ is not weakly dominated. Show why using two-person example: show it's not dominated by anything more, or by anything less.

How do we find a preferred NE? Difficult because *combining the last item with the first item shows that in every Nash equilibrium at least one player's action is weakly dominated.* (The player in question is the one who bids the same amount as bidder 1.)

Use discrete bids If bids must be in increments of ϵ, the action profile $(v_2 - \epsilon, v_2 - \epsilon, b_3, \ldots, b_n)$ with $b_i < v_i$ for $i \geq 3$ is a NE without any weakly dominated strategies. On this *ad hoc* basis we choose $(v_2, v_2, b_3, \ldots, b_n)$ with $b_i < v_i$ for $i \geq 3$ as the "distinguished" equilibriums of a first-price sealed bid auction.

Revenue equivalence! The single NE in which no player's bid is weakly dominated in a second-price sealed bid auction yields the same outcome as the distinguished equilibriums of a first-price auction.

Accident Law

Skipping this section for now!

Look at and then turn in PS 2

Game Theory, Day 9: Cancelled due to sickness

I asked the students to look at and re-do problem 59.2.

Game Theory, Day 10: Mixed strategies

Matching pennies

Solve this game (see Figure 100.1). Determine best response functions, look for Nash equilibrium. Use p and q.

Look at problem 59.2

Game Theory, Day 11: Mixed strategies

Finish problem 59.2

Hawk-Dove with mixed strategies?

This is Problem 114.1. Ag-Ag yields $(0, 0)$; Pas-Pas yields $(3, 3)$; Pas-Ag yields 1 for the Pas and 6 for the Ag. Find all the mixed strategy Nash equilibriums, including the pure strategy ones. (You get the two pure strategies, plus one symmetric mixed with pr(Ag)= $\frac{3}{4}$ and pr(Pas)= $\frac{1}{4}$.)

Game Theory, Day 12: More mixed strategies

Hawk-Dove with mixed strategies

This is Problem 114.1. Ag-Ag yields $(0,0)$; Pas-Pas yields $(3,3)$; Pas-Ag yields 1 for the Pas and 6 for the Ag. Find all the mixed strategy Nash equilibriums, including the pure strategy ones. (You get the two pure strategies, plus one symmetric mixed with pr(Ag)= $\frac{3}{4}$ and pr(Pas)= $\frac{1}{4}$.)

Relate this to road rage? Small probability of going nuts. . . .

Formalizing mixed strategies

Formal definition: strategic game with vNM preferences

Here is a formal definition of a strategic game with vNM preferences (see p. 106)

- Players
- For each player, a set of possible actions.
- For each player, preferences that can be represented by maximizing an expected payoff function. *We will come back and talk about this before spring break, but first I want to focus on the nuts and bolts; for now, then, all you need to know is that the players maximize their expected payoff.*

As before, there is no time element, but actions can be conditional

Formal definition: mixed strategy

A mixed strategy (p. 107) is a probability distribution over the player's actions. Assigning a mixed strategy α_i to each of the n players in a game generates a **mixed strategy action profile** $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. *Note that probability distributions consist of non-negative numbers that sum to one.*

Formal definition: mixed strategy Nash equilibrium

- Nash equilibrium: the idea
  - Each player's action is optimal given her beliefs about the other players' actions (theory of rational choice)
  - Each player's beliefs about other players' actions are correct!
- Nash equilibrium: more formally
  - A **mixed strategy Nash equilibrium** is an action profile $\alpha^* = (\alpha_1^*, \alpha_2^*, \ldots, \alpha_n^*)$ with the property that no player i can do better by choosing a mixed strategy different from α_i^*, given that every other player j adheres to α_j^*.

- In English, take 1: The players' actions are **mutual best responses**.
- In English, take 2: The players' actions are **second-guess-proof**.
- In English, take 3: No player can gain by deviating alone.

- Nash equilibrium: most formally
  - Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a mixed strategy action profile. Now let (α_i', α_{-i}) be the action profile where player i "deviates" by playing mixed strategy α_i' while all other players adhere to α.
  - Formally, then, if $u_i(\alpha)$ is the utility that player i gets from the mixed strategy action profile α, the mixed strategy action profile α^* is a Nash equilibrium if $u_i(\alpha^*) \geq u_i(\alpha_i, \alpha_{-i}^*)$ for all players i and all actions α_i.

Proposition 119.1

(Nash, 1950, 1951) Every strategic game with vNM preferences in which each player has finitely many actions has a mixed strategy Nash equilibrium.

Game Theory, Day 13: More mixed strategies

Results about mixed strategies

Last time we saw

Proposition 119.1

(Nash, 1950, 1951) Every strategic game with vNM preferences in which each player has finitely many actions has a mixed strategy Nash equilibrium.

Lemma 116.1

A player's expected payoff from a mixed strategy is a weighted average of her expected payoffs from each of her pure strategies, with the weights equal to her probabilities of playing those pure strategies. More formally: A player's expected payoff to the mixed strategy profile α is a weighted average of her expected payoffs to all mixed strategy profiles of the type (a_i, α_{-i}), where the weight attached to (a_i, α_{-i}) is the probability $\Pr(a_i)$ assigned to a_i by player i's mixed strategy α_i:

$$\mathrm{Exp}_i(\alpha) = \sum_{\text{pure strategies } a_i} \Pr(a_i)\mathrm{Exp}_i(a_i, \alpha_{-i}).$$

Proposition 116.2

A mixed strategy profile α^* is a MSNE if and only if, for each player i, the expected payoff, given α^*_{-i}, to every action to which α^*_i assigns positive probability is the same, and is at least as big as the expected payoff to every action to which α^*_i assigns zero probability. As a result, each player's expected payoff in an equilibrium is her expected payoff to any of her actions that she uses with positive probability.
$\Longrightarrow$ This follows because players can always deviate to pure strategies; so they must be indifferent between anything they're mixing between.

Corollary 118

Because any player who mixes must be indifferent among the different pure strategies they're mixing between, a mixed strategy Nash equilibrium in which players' mixed strategies are nondegenerate (i.e., are not also pure strategies in which they play some action with probability one) is *never* a strict Nash equilibrium.

Corollary 121

A strictly dominated action is not used with positive probability in any mixed strategy Nash equilibrium.

Note that actions can be strictly dominated by mixed strategies even when they're not dominated by any pure strategy! For example, consider the game in Figure 120.1. The strategy T is not strictly (or even weakly) dominated by either M or B, but it is not hard to see that T is indeed strictly dominated by a judicious mix of M and B. Player 1's payoff from playing T is always 1. Her expected payoff from playing M and B with probabilities p and $1-p$ is (if Player 2 plays L and R with probabilities q and $1-q$)

$$pq(4) + p(1-q)(0) + (1-p)q(0) + (1-p)(1-q)(3) = 7pq + 3 - 3p - 3q.$$

This is greater than 1 (the payoff from playing T) regardless of Player 2's choice of $q, 0 \leq q \leq 1$, if Player 1 chooses, say, $p = \frac{1}{2}$.

Game Theory, Day 14: Exam prep

Exam prep

Comments on PS 3:

1. Do not confuse best responses with beneficial deviations. Example in 80.1: if firm 1 chooses t_1 such that $h(t_1) > \frac{1}{2}$, firm 2 has no best response. Any choice of t_2 with $\frac{1}{2} \leq t_2 < t_1$ is a beneficial deviation relative to $t_2 = t_1$, but there's no best response because it's impossible to find a "smallest" ϵ for $t_2 = t_1 - \epsilon$. Example in 80.2: if person 1 chooses to devote $y_1 < 1$ to fighting, then personal 2 has no response. Any choice of y_2 with $y_2 > y_1$ is a beneficial deviation relative to $y_2 \leq y_1$, but there's no best response because it's impossible to find a "smallest" ϵ for $y_2 = y_1 + \epsilon$. *What might be going on here is a confusion about the "best response" way to find Nash equilibriums and the "direct" way to find Nash equilibriums. The fact that player 2 can gets less by choosing $y_2 = y_1 < 1$ than by choosing $y_2 = y_1 + \epsilon$ means that there is no Nash equilibrium with $y_2 = y_1 < 1$ because player 2 can gain by deviating, but it does not mean that $y_2 = y_1 + \epsilon$ is player 2's best response.*

2. Matrix and direct methods of finding all mixed strategy Nash equilibriums.

Game Theory, Day 15: First exam

Game Theory, Day 16: vNM utility; last day before Spring Break

vNM utility

Do handout from health econ.

Game Theory, Day 17: Extensive games with perfect information

An extensive game with perfect information consists of

- players
- terminal histories (complete "stories" of how the game might proceed)
- for each player, preferences over the set of terminal histories
- player function that assigns whose "move" it is at any point in the game.

Such games can be represented using a game tree.

Backward induction

Anticipate the other players' choices; think ahead and reason back.

1. Entry game
2. Do random game
3. Note that backward induction doesn't always give us an answer

Nash equilibrium

We're looking for mutual best responses. In order to determine if the players' action are mutual best responses, we need to know how a player's payoff would change if she played differently, and in order to determine this we need to know what other players would have done had the game gone differently (i.e., counterfactually).

- Example: entry game. If the challenger stays out, you need to know what the incumbent would have done.
- But it turns out to be useful to also consider what would have happened if *you'd* played differently. Do example in which player 1 chooses twice.

The punch line: we want to know each player's **strategy**, which tells us what action that player would choose at each of their choice points in the game tree. One way of thinking about this is that we're having a proxy play on our behalf, and because the proxy occasionally makes mistakes we need to specify what would happen in the event of such mistakes.

If we do this, we get the strategic form of an extensive game: have an action for each strategy in the extensive game. Conclusion: The set of Nash equilibria of any extensive game with perfect information is the set of Nash equilibria of its strategic form.

- Example: entry game. Note that there are two Nash equilibria. One involves a commitment issue.

Subgame perfect Nash equilibrium

To address commitment issues, we examine subgames and only consider strategy profiles that are Nash equilibria in each subgame: A **subgame perfect Nash equilibrium** is a strategy profile that induces a Nash equilibrium in every subgame.

- Interpretation #1: Nash equilibria come from experience playing against opponents drawn randomly from some population; SPNE come from experience playing against opponents in situations in which "trembling hands" occasionally lead to actions off the equilibrium path; such events force the players to play optimally in subgames that are off the equilibrium path.
- Interpretation #2: When each player has a unique best action at each choice point, SPNE corresponds to backward induction!

Finding SPNE using backward induction

- Start at the end of the game and work back towards the beginning.
- In the event that one player has multiple optimal moves, *separately* examine each possibility.
- Example: p. 172.

Result: Every finite extensive game with perfect information has an SPNE. Many infinite extensive games do too, as the following examples show.

Example: The rotten kid theorem

A child's action a generates income $c(a)$ for her and $p(a)$ for her parents. The child cares only about her own income level. The parents love the child so much that their utility is determined by the *smaller* of their income and their child's income; in order to maximize their utility, they can give money to (or take money from) the child after the child's choice of a is revealed.

We model this game with the child choosing a and then the parents determining how much income t they will give to the child. To solve it, use backward induction and note that the parents will choose t such that $p(a) - t = c(a) + t$, i.e., $t = \frac{1}{2}(p(a) - c(a))$. The child therefore chooses a to maximize

$$c(a) + t = c(a) + \frac{1}{2}(p(a) - c(a)) = \frac{1}{2}(p(a) + c(a)).$$

But the choice of a that maximizes this also maximizes $p(a) + c(a)$, i.e., the "rotten kid" makes a choice that maximizes family income!

Game Theory, Day 18: Ultimatum and Stackelberg

The ultimatum game

One round

Player 1 proposes a division of \$8 between himself and Player 2. Player 2 either accepts or rejects. In subgames in which Player 1 offers Player 2 a positive amount, Player 2's unique optimal action is to accept; if Player 1 offers Player 2 zero, both accepting and rejecting are optimal for Player 2. In the latter case, there are no Nash equilibria because there is no smallest ϵ. In the former case, we get that the unique SPNE (with infinite divisibility) is for Player 1 to offer Player 2 \$0 and for Player 2 to accept.

Conclusion: The only SPNE results in the offering player getting 100% of the payoff and the other player getting nothing. But this is not what we see experimentally! (Sometimes Player 1 offer more than zero, and sometimes Player 2 refuses a non-zero offer.)

The melting ice cream pie game, two rounds

Same as above, only now if Player 2 rejects then Player 2 gets to propose a division of \$4 between herself and Player 1. Player 1 either accepts or rejects, and if he rejects then the money is gone.

We know from before that the only SPNE in the second stage is for Player 2 to offer zero and for Player 1 to accept. Considering the first round, then, we need to split Player 1's options into three sets. If Player 1 offers Player 2 less than \$4, Player 2's best response is to reject the offer and move to the second stage; if Player 1 offers Player 2 more than \$4, Player 2 should accept the offer; and if Player 1 offers Player 2 exactly \$4, both accepting and rejecting are best responses for Player 2. If Player 2 accepts a \$4 offer, then the SPNE is for Player 1 to offer \$4 and for Player 2 to accept. If Player 2 rejects a \$4 offer, then there is no SPNE because there is no smallest ϵ.

Conclusion: The only SPNE results in the players splitting the payoff.

The melting ice cream pie game, three rounds

Same as above, only now if Player 1 rejects Player 2's counter-offer then he gets to make a third (and final) proposal for dividing \$2; as usual, if Player 2 rejects this offer the money is gone.

We know from before that the only SPNE in the second stage is for Player 2 to offer a 50-50 split of the \$4 that will be available at that time. Considering the first round, then, we need to split Player 1's options into three sets. If Player 1 offers Player 2 less than \$2, Player 2's best response is to reject the offer and move to the second stage; if Player 1 offers Player 2 more than \$2, Player 2 should accept the offer; and if Player 1 offers Player 2 exactly \$2, both accepting and rejecting are best responses for Player 2. If Player 2 accepts a \$2 offer, then the SPNE is for Player 1 to offer \$2 and for Player 2 to accept. If Player 2 rejects a \$2 offer, then there is no SPNE because there is no smallest ϵ.

Stackelberg

Imagine an industry with no fixed costs and constant marginal costs of $C(q) = 2q$; the demand curve for the industry's output is $q = 10 - p$.

Monopoly

The monopolist chooses p and q to maximize $\pi = pq - C(q) = pq - 2q$ subject to the constraint $q = 10 - p$. Equivalently, the monopolist chooses q to maximize $\pi = (10-q)q - 2q = 8q - q^2$. Setting a derivative equal to zero gives us $8 - 2q = 0$, meaning that the monopolist will choose $q = 4$, $p = 10 - q = 6$, and get profits of $\pi = pq - C(q) = 16$.

Bertrand duopoly

Imagine that a second firm enters the market and engages in Bertrand (price) competition with the first firm. (This is price competition: if both firms charge the same price then they split the market; otherwise the low-price firm gets the whole market.) We know that the only NE is for both firms to charge a price of $p = c$. (First, we know that this is a NE. Second, there can't be a NE with any price less than c because then a low-price firm is making negative profits and can gain by deviating to $p = c$. Third, there can't be a NE with both prices greater than c because a high-price firm can gain by deviating to ϵ less than the other firm. Fourth, there can't be a NE with one firm charging c and the other firm charging more than c because the low-price firm can gain by deviating to $c + \epsilon$. So we get $q = 8$, $p = 2$, and profits of zero for both firms.

Cournot Duopoly

Imagine that a second firm enters the market and engages in Cournot (quantity) competition with the first firm. The demand curve is still $q = 10 - p$, but now $q = q_1 + q_2$, i.e., the market output is the sum of each firm's output. We can transform the demand curve $q_1 + q_2 = 10 - p$ into the inverse demand curve, $p = 10 - q_1 - q_2$.

Now, firm 1 chooses q_1 to maximize $\pi_1 = pq_1 - C(q_1) = (10 - q_1 - q_2)q_1 - 2q_1 = (8 - q_2)q_1 - q_1^2$. Setting a derivative equal to zero gives us $8 - q_2 - 2q_1 = 0$, which rearranges as $q_1 = 4 - .5q_2$. This is the best response function for firm 1: given q_2, it specifies the choice of q_1 that maximizes firm 1's profits.

Since the problem is symmetric, firm 2's best response function is $q_2 = 4 - .5q_1$. Solving these simultaneous to find the Cournot solution yields

$$q_1 = 4 - .5(4 - .5q_1) \Longrightarrow .75q_1 = 2 \Longrightarrow q_1 = \frac{8}{3} \approx 2.67.$$

We get the same result for q_2, so the market price will be $10-q_1-q_2 = \frac{14}{3} \approx 4.67$. Each firm will earn profits of

$$pq_i - 2q_i = \frac{14}{3} \cdot \frac{8}{3} - 2 \cdot \frac{8}{3} = \frac{64}{9} \approx 7.11,$$

so industry profits will be about $2(7.11) = 14.22$.

Stackelberg leader-follower duopoly

Imagine that a second firm enters the market and engages in Stackelberg competition with the first firm. (In this model, also called a **leader-follower model**, firm 1 plays first by choosing q_1; after seeing this choice, firm 2 then chooses q_2.

Using backward induction, we begin with firm 2's choice. It has to choose q_2 to maximize $\pi_2 = pq_2 - 2q_2 = (8 - q_1)q_2 - q_2^2$., *which is the same problem as before*. As before, then, we get firm 2's best response function, $q_2 = 4 - .5q_1$.

The difference comes with firm 1's choice. It chooses q_1 to maximize $\pi_1 = pq_1 - C(q_1) = (10-q_1-q_2)q_1 - 2q_1 = (8-q_2)q_1 - q_1^2$, but now *firm 1 doesn't take q_2 as a constant*: firm 1 can look ahead (using the ideas of backward induction or subgame perfection) and anticipate that firm 2's choice of q_2 is going to depend on q_1, i.e., that $q_2 = f(q_1)$.

In particular, firm 1 knows that firm 2 is going to choose $q_2 = 4 - .5q_1$. So firm 1 plugs this into its profit function:

$$\pi_1 = (8-q_2)q_1 - q_1^2 = (8-(4-.5q_1))q_1 - q_1^2 = (4+.5q_1)q_1 - q_1^2 = 4q_1 + .5q_1^2 - q_1^2 = 4q_1 - .5q_1^2.$$

Taking a derivative and setting it equal to zero we get $4 - q_1 = 0$, i.e., $q_1 = 4$. We can then use firm 2's best response function to find $q_2 = 2$. We therefore have $q_1 = 4$, $q_2 = 2$, $p = 4$, and profits of $\pi_1 = 8$ and $\pi_2 = 4$. Note that firm 1 chooses the monopoly level of output here; this is just a coincidence! Not a coincidence is that profits are lower than in a Cournot situation (and, of course, lower than in monopoly). Relative to the Cournot outcome, firm 1 can steal some of firm 2's profits by increasing its output.

Game Theory, Day 19: No class because of Whitman Undergraduate Conference

Game Theory, Day 20: Stackelberg example with fixed costs

I tried to follow exercise 191.1, but ran into trouble; a partial write-up is in stackelberg.tex. (See exercise 59.2 and the solution in PS2.) If you do this again, use $f = 10$ for an example of a "natural monopoly", $f = 4$ as an example of an "unnatural monopoly" (i.e., where firm 1 overproduces to keep firm 2 out), and $f = 1$ (I think) for an example where we get the no-fixed-cost Stackelberg result.

Game Theory, Day 21: Repeated games

A **repeated game** is an extensive-move game in which players play the same **stage game** a finite (or infinite) number of times. We are going to study a repeated game in which the stage game is the Prisoners' Dilemma game with a payoff from cooperative of $(2, 2)$, a payoff from mutual defection of $(1, 1)$, and a combo payoff of 3 for the defector and 0 for the cooperator.

Preferences

We assume that each player tries to maximize a **discounted sum** of their payoffs from the various stage games:

$$\pi = \pi_1 + \delta\pi_2 + \delta^2\pi_3 + \ldots$$

for some $0 < \delta < 1$. Stories about why this might make sense:

- Players are impatient.
- The game takes time and players can lend or borrow money at some interest rate r. (In this case, $\delta = \frac{1}{1-r}$. High values of δ correspond to low values of r, and vice versa.)
- There is some probability p that the game will end (or that one of the players will die) at the end of each stage. (In this case, $\delta = 1 - p$. Again, high values of δ correspond to low values of p, and vice versa.) Notice that in this case the probability of the game lasting more than n stages is δ^n, which gets small very quickly.

What's common in all these stories is that players value the present more than the future.

Two handy results from algebra are that, for $0 < \delta < 1$,

$$x + \delta x + \delta^2 x + \ldots + \delta^n x = x \left[\frac{1 - \delta^{n+1}}{1 - \delta} \right]$$

and

$$x + \delta x + \delta^2 x + \ldots = \frac{x}{1 - \delta}.$$

Finitely repeated games

Using backward induction we can see that there will never be any cooperation in a finitely repeated Prisoners' Dilemma Game. The only SPNE is for each player to always play D, and in fact the only NE result in an outcome of (D, D) in every period.

Infinitely repeated games

Claim 1 *Strategies in which the outcome is (D, D) in every period form a NE in an infinitely repeated game, and strategies in which both players always play D generates an SPNE.*

Claim 2 *If the players are "patient enough" (i.e., if δ is "big enough"), there are NEs in which the outcome is (C, C) in every period.*

Consider, for example, the "trigger" strategy:

- Play C if neither player has ever played D. (In other words, play C in the first stage, and continue to play C as long as (C, C) has been the outcome in all previous rounds.)
- Play D otherwise.

If the other player adopts this trigger strategy, what is your payoff from adopting the same trigger strategy? Well, you'll both play C forever, and each of you will get a payoff of $\frac{2}{1-\delta}$.

Is this your best response? Can you gain by deviating alone? Well, the best way to deviate is to play D in the nth stage game, and then (since the other player will play D thereafter), to play D thereafter. This deviation is attractive if and only if deviating in the first stage is attractive (explain why), and if you deviate in the first stage then your payoff will be

$$3 + 1\delta + 1\delta^2 + \ldots = 3 + \frac{\delta}{1-\delta} = \frac{3-2\delta}{1-\delta}.$$

You can gain from this deviation if and only if

$$\frac{3-2\delta}{1-\delta} > \frac{2}{1-\delta},$$

i.e., if and only if $3 - 2\delta > 2$, i.e., if and only if $\delta < \frac{1}{2}$.

We conclude that you *cannot* gain by deviating alone if $\delta \geq \frac{1}{2}$, which proves that mutual trigger strategies form a NE in this infinitely repeated PD with $\delta \geq \frac{1}{2}$.

Claim 3 *These mutual trigger strategies also form a SPNE.*

In all subgames without any history of D being played, the mutual trigger strategies form a NE. And in all subgames in which D has previously been played, both players will forever play D, and we know from before that that is also a NE.

Claim 4 *Be careful with NE and SPNE!*

Consider this "alternative trigger" strategy:

- Play C if the other player has never played D.
- Play D otherwise.

This is a NE (both players play C forever), but it is *not* a SPNE. Consider, for example, the subgame beginning after the play of (C, D) in the first stage. In this subgame, player 1 will play D forever; but player 2 will play C in the first stage! This is clearly not a best response to player 1's strategy.

Claim 5 *You can get all kinds of weird NE (and SPNE) with trigger strategies.*

For example, you can think of the "always play D" strategy as a sort of trigger strategy: You've pissed me off, so I'm going to always play D. Here's another weird trigger strategy:

- Play C in odd-numbered stages and D in even-numbered stages as long as neither player has ever played D in an odd-numbered stage or C in an even-numbered stage.
- Play D otherwise.

With a high enough δ, this is a SPNE. (Have the class do this as a problem?) Maybe do the folk theorem from here?

Game Theory, Day 22: Tit-for-tat

Claim 6 *Tit-for-tat (play C in the first round, and thereafter play whatever your opponent played in the previous round) is a NE as long as δ is big enough.*

If you both play tit-for-tat, you play (C, C) forever and your payoff is $\frac{2}{1-\delta}$. Note that if you can't successfully deviate in round 1 then you can't successfully deviate ever, which means that we can limit our attention to deviations in round 1. So: consider the deviation where you play D in round 1.

What kind of deviations might succeed? Well, you could just play D forever. Or you could play D for a while and then switch back to C. But if it won't pay to switch back to C in round 2 then it won't pay to switch back to C ever, so we can limit our attention here to deviations in which you play D in round 1 and then C in round 2. And this brings us back to the game's starting point. So there are two deviations we need to consider: the one in which you "play nasty" (always play D) and the one in which you "pull a fast one" (play D and then play C forever).

If you always play D then your payoff is

$$3 + (1)\delta + (1)\delta^2 + \ldots = 3 + \frac{\delta}{1-\delta} = \frac{3-2\delta}{1-\delta}.$$

This is not a successful deviation as long as $\frac{2}{1-\delta} \geq \frac{3-2\delta}{1-\delta}$, i.e., as long as $2 \geq 3-2\delta$, i.e., as long as $\delta \geq \frac{1}{2}$.

If you play D and then C forever then your payoff is $3+0+(2)\delta^2(2)\delta^3+\ldots$, which is the same as your tit-for-tat payoff except that you get 3 (instead of 2) in round 1 and 0 (instead of 2) in round 2. This is not a successful deviation as long as $2 + 2\delta \geq 3$, i.e., as long as $\delta \geq \frac{1}{2}$.

So: As long as δ is big enough, mutual tit-for-tat generates a Nash equilibrium.

Claim 7 *It's much harder for tit-for-tat to be an SPNE.*

Consider a subgame starting immediately after a round of (D, D). Tit-for-tat leads to an outcome of (D, D) for the rest of the game, generating a payoff in that subgame of $1 + (1)\delta + \ldots = \frac{1}{1-\delta}$. If player 1 deviates to C forever, then player 1's payoff will be $0 + 2\delta + 2\delta^2 + \ldots = \frac{2\delta}{1-\delta}$. Tit-for-tat is at least as good as this alternative as long as $1 \geq 2\delta$, i.e., as long as $\delta \leq \frac{1}{2}$.

Game Theory, Day 23: Evolutionary Game Theory

Preface: This is new material for me, so I'm working as hard as you are to understand it.

Basic ideas and differences with "regular" game theory

"Regular" game theory studies interactions between optimizing individuals. In contrast, evolutionary game theory is better thought of as studying interactions between optimizing *genes*; the individuals produced by those genes are simply programmed to act however the genes specify; they may not be optimizing at all. This requires a change in perspective; as the famous saying has it, instead of thinking about genes as being people's way of making more people, you need to think of people as being genes way of making more genes. So: the "strategies" in evolutionary game theory are genetic strategies, e.g., "be aggressive" or "be friendly". And the payoffs are measures of reproductive fitness, i.e., measures of how many children (either people or genes) are produced by a given strategy.

We begin by studying **monomorphic pure strategies**; "pure strategies" indicates that we are not (yet) looking at mixed strategies (i.e., there is no randomness), and "monomorphic" ("one form") indicates that the population is essentially homogenous: everyone is the same. For now we are also assuming that we have asexual reproduction.

Of course, in a population in which all the genes are the same, there's no evolution. Evolution happens because random mutations will occasionally occur in an otherwise-homogenous population. The vast majority of the time these mutations are either harmful or irrelevant. But in some cases these mutations are beneficial, the mutants will "invade" the population, and—lo and behold—we get opposable thumbs.

The idea of an **evolutionary equilibrium** is of a gene make-up that repels invasion: the appearance of a small group of mutants fails to have a long-run impact. In other words, mutants die.

Note that there is a difference between a gene make-up that is *invasion-repellent* and one that is *invasion-resistant*. The latter term might be applied in cases in which mutants simply *fail to thrive*, i.e., they do no better than the main form. The former term, which we focus on, applies to cases in which mutants die.

What do we mean by "die"? Well, we might reasonably mean that their presence (in terms of percentages) declines over time. In other words, mutants don't reproduce as well as regulars. (If growth is high, it is possible that mutants wouldn't "die out", but would still become increasingly rare.)

So the basic idea is to see how an essentially homogenous population is affected by the entrance of a small number of mutants. We will assume that the percentage of mutants is some small number ϵ. And we will assume that the population (both regular and mutant) is large enough that any individual (whether regular or mutant) encounters a mutant with probability ϵ and encounters a non-mutant with probability $1 - \epsilon$.

Examples

Player 2

Player 1		X	Y
	X	2,2	0,0
	Y	0,0	1,1

Figure 1: Both (X,X) and (Y,Y) are evolutionarily stable

Player 2

Player 1		X	Y
	X	2,2	0,0
	Y	0,0	0,0

Figure 2: (X,X) is evolutionarily stable, but (Y,Y) is not

So we can see that Nash equilibrium is perhaps necessary but not sufficient for evolutionary stability.

Examples

I asked the class to try to apply the Prisoners' Dilemma idea to evolutionary game theory. Fun!

Game Theory, Day 24: More Evolutionary Game Theory

Formal definitions

Let $u(a, b)$ denote the payoff from playing a against an opponent playing b. Then the action a^* is **evolutionarily stable** if there exists $\bar{\epsilon} > 0$ such that

$$(1-\epsilon)u(a^*, a^*) + \epsilon u(a^*, b) > (1-\epsilon)u(b, a^*) + \epsilon u(b, b)$$

for all $b \neq a^*$ and all ϵ with $0 < \epsilon < \bar{\epsilon}$.

Claim 8 *If $u(a^*, a^*) > u(b, a^*)$ for all $b \neq a$, then a^* is evolutionarily stable.*

Note that this implies that if (a^*, a^*) is a *strict* Nash equilibrium then a^* is evolutionarily stable.

Claim 9 *If $u(b, a^*) > u(a^*, a^*)$ for some $b \neq a$, then a^* is* not *evolutionarily stable.*

Note that this implies that if (a^*, a^*) is *not* a Nash equilibrium then a^* is not evolutionarily stable.

So what if (a^*, a^*) is a Nash equilibrium but not a strict Nash equilibrium? Well, we can define an evolutionary stable action without using ϵ; it's an action such that

- (a^*, a^*) is a Nash equilibrium, and
- $u(b, b) < u(a^*, b)$ for every action $b \neq a^*$ that is a best response to a^*.

Examples

The games above; the Prisoners' Dilemma game (which also shows that evolutionary stability does not mean "optimal behavior" on the part of the species, e.g., antlers or reproductive proportions), BoS (see p. 403), which has no ESS. In the Hawk-Dove game, show that $v > c$ or $v = c$ implies a unique ESS, and that $v < c$ implies no ESS in pure strategies. (Do examples with $(v = 20, c = 10)$, $(v = 20, c = 20)$, and $(v = 20, c = 40)$.

Mixed strategies

Definition (see p. 400): An **evolutionarily stable strategy (ESS)** is a mixed strategy α^*

- (α^*, α^*) is a Nash equilibrium, and
- $u(\alpha^*, \beta) > u(\beta, \beta)$ for every action $\beta \neq \alpha^*$ that is a best response to α^*.

Even though the relationship between evolutionary stability in pure strategies and evolutionary stability in mixed strategies looks very similar to the relationship between Nash equilibrium in pure strategies and Nash equilibrium in mixed strategies. But there's a difference: any Nash equilibrium in pure strategies is also a mixed strategy Nash equilibrium, but pure-strategy evolutionary stability does not guarantee mixed-strategy evolutionary stability. See example, p. 402.

Game Theory, Day 25: More Evolutionary Game Theory

Mixed strategies

Definition (see p. 400): An **evolutionarily stable strategy (ESS)** is a mixed strategy α^*

- (α^*, α^*) is a Nash equilibrium, and
- $u(\alpha^*, \beta) > u(\beta, \beta)$ for every action $\beta \neq \alpha^*$ that is a best response to α^*.

BoS (403.1)

In BoS, player 1 chooses p to maximize

$$pq(0)+p(1-q)(2)+(1-p)(q)(1)+(1-p)(1-q)(0) = 2p-2pq+q-pq = 2p+q-3pq.$$

We either have a corner solution (either $p = 0$ or $p = 1$) or an interior solution with $2 - 3q = 0$, i.e., with $q = \frac{2}{3}$. So player 1's best response function is to choose $p = 0$ if $q > \frac{2}{3}$, to choose $p = 1$ if $q < \frac{2}{3}$, and to choose any $p, 0 \leq p \leq 1$, if $q = \frac{2}{3}$.

We're looking for a symmetric equilibrium, i.e., with $p = q$. The corner solutions don't work because the best response to $p = 1$ is $q = 0$ and the best response to $p = 0$ is $q = 1$. The interior solution does work: we can solve player 2's problem to see that any $q, 0 \leq q \leq 1$, is a best response to $p = \frac{2}{3}$.

So the only NE with $p = q$ is $p = q = \frac{2}{3}$. This is therefore the only candidate for an ESS. To see if it's an ESS, we have to consider all actions $p \neq \frac{2}{3}$ that are a best response to $\frac{2}{3}$. But this includes all actions $p \neq \frac{2}{3}$! So: we need to consider all actions $p \neq \frac{2}{3}$ and see if a mutant with $p \neq \frac{2}{3}$ can do better than $\frac{2}{3}$ against another mutant with $p \neq \frac{2}{3}$.

Well, the payoff for a mutant against a mutant is

$$p^2(0) + p(1-p)(2) + (1-p)(p)(1) + (1-p)^2(0) = 3p(1-p) = 3p - 3p^2.$$

And the payoff for a normal type playing $\frac{2}{3}$ against a mutant is

$$\frac{2}{3}(p)(0) + \frac{2}{3}(1-p)(2) + \frac{1}{3}(p)(1) + \frac{1}{3}(1-p)(0) = \frac{4}{3}(1-p) + \frac{1}{3}(p) = \frac{4}{3} - p.$$

The payoff for the normal type is strictly greater if

$$\frac{4}{3} - p > 3p - 3p^2 \Longrightarrow 3p^2 - 4p + \frac{4}{3} \Longrightarrow 3\left(p - \frac{2}{3}\right)^2 > 0.$$

This is always true for $p \neq \frac{2}{3}$, so we conclude that $\frac{2}{3}$ is indeed an ESS.

Polymorphic steady states

We can reinterpret mixed strategies as representing **polymorphic steady states**, i.e., states in which different percentages of the population follow different strategies (or have different genotypes). In this case, the idea of an ESS matches up with the idea of **stability of percentages**, meaning that mutations that slightly change the percentages have no long run impact because the percentages return to their steady-state values.

Coordination game (404.1)

In the coordination game, player 1 chooses p to maximize

$$pq(2)+p(1-q)(0)+(1-p)(q)(0)+(1-p)(1-q)(1) = 2pq+1-p-q+pq = 3pq+1-p-q.$$

We either have a corner solution (either $p = 0$ or $p = 1$) or an interior solution with $3q - 1 = 0$, i.e., with $q = \frac{1}{3}$. So player 1's best response function is to choose $p = 1$ if $q > \frac{1}{3}$, to choose $p = 0$ if $q < \frac{1}{3}$, and to choose any $p, 0 \leq p \leq 1$, if $q = \frac{1}{3}$.

We're looking for a symmetric equilibrium, i.e., with $p = q$. The corner solutions $p = q = 1$ and $p = q = 0$ are both NE, and in fact they are both strict NE, so both are ESS. There is also an interior solution: we can solve player 2's problem to see that any $q, 0 \leq q \leq 1$, is a best response to $p = \frac{1}{3}$.

Other than the corner solutions, then, $p = q = \frac{1}{3}$ is the only symmetric NE and therefore is the only candidate for an ESS. To see if it's an ESS, we have to consider all actions $p \neq \frac{1}{3}$ that are a best response to $\frac{1}{3}$. But this includes all actions $p \neq \frac{1}{3}$! So: we need to consider all actions $p \neq \frac{1}{3}$ and see if a mutant with $p \neq \frac{1}{3}$ can do better than $\frac{1}{3}$ against another mutant with $p \neq \frac{1}{3}$.

Well, the payoff for a mutant against a mutant is

$$p^2(2) + p(1-p)(0) + (1-p)(p)(0) + (1-p)^2(1) = 2p^2 + (1-p)^2.$$

And the payoff for a normal type playing $\frac{1}{3}$ against a mutant is

$$\frac{1}{3}(p)(2) + \frac{1}{3}(1-p)(0) + \frac{2}{3}(p)(0) + \frac{2}{3}(1-p)(1) = \frac{2}{3}p + \frac{2}{3}(1-p) = \frac{2}{3}.$$

The payoff for the normal type is strictly greater if $\frac{2}{3} > 2p^2 + (1-p)^2$ for all $p \neq \frac{1}{3}$. This is not true, e.g., for $p = 1$ or $p = 0$, so we conclude that $\frac{1}{3}$ is not an ESS.

Hawk-Dove(404.2, see figure 399.1)

In Hawk-Dove, if $v > c$ then A is a strictly dominant strategy and therefore A is the only symmetric ESS.

In Hawk-Dove, player 1 chooses p to maximize

$$pq \cdot \frac{1}{2}(v-c) + p(1-q)(v) + (1-p)(q)(0) + (1-p)(1-q) \cdot \frac{1}{2}v.$$

We either have a corner solution (either $p = 0$ or $p = 1$) or an interior solution with

$$q \cdot \frac{1}{2}(v-c) + (1-q)(v) - (1-q) \cdot \frac{1}{2}v = \frac{-qc}{2} + v - \frac{1}{2}v = \frac{-qc+v}{2} = 0,$$

i.e., with $q = \frac{v}{c}$. So player 1's best response function is to choose $p = 0$ if $q > \frac{v}{c}$, to choose $p = 1$ if $q < \frac{v}{c}$, and to choose any $p, 0 \leq p \leq 1$, if $q = \frac{v}{c}$.

We're looking for a symmetric equilibrium, i.e., with $p = q$. The corner solutions don't work because the best response to $p = 1$ is $q = 0$ and the best response to $p = 0$ is $q = 1$. The interior solution does work for $v \leq c$: we can solve player 2's problem to see that any $q, 0 \leq q \leq 1$, is a best response to $p = \frac{v}{c}$.

So the only NE with $p = q$ is $p = q = \frac{v}{c}$. This is therefore the only candidate for an ESS. To see if it's an ESS, we have to consider all actions $p \neq \frac{v}{c}$ that are a best response to $\frac{v}{c}$. But this includes all actions $p \neq \frac{v}{c}$! So: we need to consider all actions $p \neq \frac{v}{c}$ and see if a mutant with $p \neq \frac{v}{c}$ can do better than $\frac{v}{c}$ against another mutant with $p \neq \frac{v}{c}$.

Well, the payoff for a mutant against a mutant is

$$p^2 \cdot \frac{1}{2}(v-c) + p(1-p)(v) + (1-p)(p)(0) + (1-p)^2 \cdot \frac{1}{2}v,$$

which simplifies to

$$\frac{1}{2}[p^2(v-c) + 2p(1-p)v + (1-p)^2 v] = \frac{-p^2 c}{2} + \frac{1}{2}(v)[p^2 + 2(p)(1-p) + (1-p)^2],$$

which simplifies to

$$\frac{-p^2 c}{2} + \frac{1}{2}(v)[p + (1-p)]^2 = \frac{-p^2 c}{2} + \frac{1}{2}(v) = \frac{v - p^2 c}{2}.$$

And the payoff for a normal type playing $\frac{v}{c}$ against a mutant is

$$\frac{v}{c}(p) \cdot \frac{1}{2}(v-c) + \frac{v}{c}(1-p)(v) + \frac{1-v}{c}(p)(0) + \frac{1-v}{c}(1-p) \cdot \frac{1}{2}v,$$

which simplifies to

$$\frac{v}{2c}\left[(p)(v-c) + 2(1-p)(v) + (1-v)(1-p)\right].$$

Ugly! Anyway, it turns out that the payoff for the normal type is strictly greater if $p \neq \frac{v}{c}$, so we conclude that $\frac{v}{c}$ is indeed an ESS. The higher c is relative to v, the less likely there will be a fight.

Game Theory, Day 26: More Evolutionary Game Theory

Polymorphic steady states

We can reinterpret mixed strategies as representing **polymorphic steady states**, i.e., states in which different percentages of the population follow different strategies (or have different genotypes). In this case, the idea of an ESS matches up with the idea of **stability of percentages**, meaning that mutations that slightly change the percentages have no long run impact because the percentages return to their steady-state values.

No ESS

In a Rock-Paper-Scissors game with $\gamma > 0$ (see 406.1), the only NE is the mixed strategy NE in which each player plays each option with probability $\frac{1}{3}$. But this is not an ESS: anything is a best response to this, and mutants who play pure strategies do better against similar mutants than do the dominant types. See example with side-blotched lizards, p. 407.

Asymmetric games

Consider the asymmetric version of Hawk-Dove in 409.1. This is an example of what is called a **contest game**. We can study this game via a symmetric game in which players choose their strategies conditionally, i.e., based on whether they're the owner or the intruder.

Claim 10 *A mixed strategy pair* (α_1, α_2) *is a mixed strategy NE of* C *if and only if* $((\alpha_1, \alpha_2), (\alpha_1, \alpha_2))$ *is a NE of* G*, and if* (α_1, α_2) *is a strict Nash equilibrium of* C *then* $((\alpha_1, \alpha_2), (\alpha_1, \alpha_2))$ *is a strict NE of* G*.*

Claim 11 *A conditional strategy is evolutionarily stable if and only if it is a strict (and hence pure) NE of the contest game* C*.*

So we get a bourgeois strategy and a paradoxical strategy as being evolutionarily stable.

Sex ratios

Consider a stable population with a fraction p of males (and therefore $1 - p$ of females). This is possible if offspring are males with probability p and females with probability $1 - p$. If each female averages n offspring, then each male averages $\frac{1-p}{p}n$ offspring.

Now consider a mutant dominant gene that produces male and female offspring, each with probability .5. Then a mutant female produces $\frac{n}{2}$ male children and $\frac{n}{2}$ female children, meaning that the number of grandchildren for the mutant female is

$$\frac{n}{2}\frac{1-p}{p}n + \frac{n}{2}n = \frac{n^2}{2p}.$$

A normal female, on the other hand, produces pn male children and $(1-p)n$ female children, meaning that the number of grandchildren for the normal female is

$$(pn)\frac{1-p}{p}n + [(1-p)n]n = 2n^2(1-p).$$

Comparing these we see that mutants have more grandchildren if $\frac{1}{2p} > 2(1-p)$, i.e., if $4p(1-p) < 1$. This is true for all $p \neq \frac{1}{2}$, so we see that nothing else is evolutionarily stable.

Game Theory Review: Topics Covered

Final exam currently scheduled for 9-11am on Wednesday May 18.

Dominant strategies Strictly and weakly dominant strategies; strictly and weakly dominated strategies.

Nash equilibrium of strategic games Describing strategies and Nash equilibrium fully; underlining and/or strike-out methods of finding Nash equilibrium; strict Nash equilibrium.

Best response functions How to construct them; relationship to Nash equilibrium.

Oligopoly models Cournot (quantity competition), Bertrand (price competition), Stackelberg (leader-follower).

Auctions Different kinds of auctions, strategies, and Nash equilibriums.

Mixed strategy Nash equilibrium How to find them.

Extensive move games Game trees and backward induction.

Subgame perfect Nash equilibrium Describing them *fully* (i.e., describing what the players will do everywhere in the game treee), understanding connection to backward induction.

Finitely repeated games Solving using backward induction/SPNE.

Infinitely repeated games Describing trigger strategies fully, finding critical values of δ that support cooperation.

Evolutionary game theory Definition of evolutionary stability, pure and mixed strategy evolutionarily stable strategies. *No asymmetric contests.*