Lots of Math

These notes review the derivation of supply curves for individual firms and demand curves for individual consumers. We begin, however, with a simpler problem: cost minimization.

1 Cost Minimization

For individuals, the cost minimization problem is to achieve a specified level of utility (say, $U = 10$) at least cost. For firms, the cost minimization problem is to produce a specified amount of output (say, $Y = 10$) at least cost. *These problems are identical: if you like, you can think of the individual as a firm whose "product" is utility, or of the firm as an individual whose "utility" depends on output.* We will reinforce this connection by using examples with similar notation: the individual we will consider gets utility from drinking lattes (L) and eating cake (K); the firm we will consider produces output from inputs of labor (L) and capital (K).

Utility Functions and Indifference Curves

Recall **utility functions** and **indifference curves**: if the individual's utility function is $U(L, K) = L^{\frac{1}{2}} K^{\frac{1}{2}}$ (an example of a **Cobb-Douglas utility function**[1]), then the indifference curve corresponding to a utility level of, say, 2 is the set of all consumption bundles that provide the individual with a utility of 2.

In our example, the indifference curve corresponding to a utility level of 2 contains the points $(L = 1, K = 4)$, $(L = 4, K = 1)$, and $(L = 2, K = 2)$. The equation for this indifference curve is $L^{\frac{1}{2}} K^{\frac{1}{2}} = 2$, which we can rewrite as $LK = 4$ or $K = 4L^{-1}$. The slope of this indifference curve, $\frac{dK}{dL} = -4L^{-2}$, measures the **marginal rate of substitution (MRS)** between lattes and cake: an individual with a utility level of 2 who currently has L lattes and K pieces of cake would be willing to trade up to $4L^{-2}$ pieces of cake in order to gain an extra latte. Such a substitution would leave the individual on the same indifference curve, and therefore with the same utility.

An important result that will be useful later is that the slope of the indifference curve (i.e., the marginal rate of substitution) can also be written as

$$\text{MRS} = -\frac{\frac{\partial U}{\partial L}}{\frac{\partial U}{\partial K}}.$$

[1] The general form of a Cobb-Douglas utility function is $U = L^{\alpha} K^{\beta}$ where α and β are positive constants.

Here the numerator is the **marginal utility of lattes** ($MU_L = \frac{\partial U}{\partial L}$), the extra utility the individual would get from an additional latte. The denominator is the **marginal utility of cake** ($MU_K = \frac{\partial U}{\partial K}$), the extra utility the individual would get from an additional piece of cake. Intuitively, the slope of the indifference curve tells us the maximum amount of cake the individual is willing to give up in order to receive one more latte. Since one more latte gives the individual MU_L extra utility, the amount of cake the individual should be willing to give up in order to get an additional latte is K such that $MU_K \cdot K = MU_L$, i.e., $K = \frac{MU_L}{MU_K}$. (For example, if the marginal utility of lattes is 3 and the marginal utility of cake is 1, the individual should be willing to give up 3 pieces of cake to get one more latte.) It follows that the slope of the indifference curve is

$$\text{MRS} = -\frac{MU_L}{MU_K} = -\frac{\frac{\partial U}{\partial L}}{\frac{\partial U}{\partial K}}.$$

Production Functions and Isoquants

Firms have structures analogous to utility functions and indifference curves; these are called **production functions** and **isoquants**. Given inputs of labor (L) and capital (K), the production function $f(L, K)$ describes the quantity of output that can be produced from these inputs. If the firm's production function is $Y = L^{\frac{1}{2}} K^{\frac{1}{2}}$ (an example of a **Cobb-Douglas production function**), then the **isoquant** corresponding to an output level of, say, 2 is the set of all input bundles that the firm can use to produce 2 units of output.

In our example, the isoquant corresponding to an output level of 2 contains the points $(L = 1, K = 4)$, $(L = 4, K = 1)$, and $(L = 2, K = 2)$. The equation for this isoquant is $L^{\frac{1}{2}} K^{\frac{1}{2}} = 2$, which we can rewrite as $LK = 4$ or $K = 4L^{-1}$. The slope of this isoquant, $\frac{dK}{dL} = -4L^{-2}$, measures the **marginal rate of technical substitution (MRTS)** between labor and capital: a firm with an output target of 2 which currently has L units of labor and K units of capital would be willing to trade up to $4L^{-2}$ units of capital in order to gain an extra unit of labor. Such a substitution would leave the firm on the same isoquant, and therefore with the same output.

An important result that will be useful later is that the slope of the isoquant can also be written as

$$\text{MRTS} = -\frac{\frac{\partial f}{\partial L}}{\frac{\partial f}{\partial K}}.$$

Here the numerator is the **marginal product of labor** ($MP_L = \frac{\partial f}{\partial L}$), the extra output the firm would get from an additional unit of labor. The denominator is the **marginal product of capital** ($MP_K = \frac{\partial f}{\partial K}$), the extra output the firm would get from an additional unit of capital. Intuitively, the slope of the isoquant tells us the maximum amount of capital the firm is willing to give up in order to get one more unit of labor. Since one unit of labor allows the firm to produce MP_L extra units of output, the amount of capital the firm should be willing to give up in order to get an additional unit of labor is K such that $MP_K \cdot K = MP_L$, i.e., $K = \frac{MP_L}{MP_K}$. (For example, if the marginal product of labor is 3 and the marginal product of capital is 1, the firm should be willing to give up 3 units of capital to get one more unit of labor.) It follows that the slope of the isoquant is

$$\text{MRTS} = -\frac{MP_L}{MP_K} = -\frac{\frac{\partial f}{\partial L}}{\frac{\partial f}{\partial K}}.$$

The Cost Function

If lattes and cake (or labor and capital) have unit prices of p_L and p_K, respectively, then the total cost of purchasing L units of one and K units of the other is

$$C(L, K) = p_L L + p_K K.$$

The cost minimization problem for the individual is to choose L and K to minimize the cost necessary to reach a specified utility level (say, $U = 2$). The cost minimization problem for the firm is to choose L and K to minimize the cost necessary to reach a specified output level (say, $Y = 2$). Mathematically, the individual wants to choose L and K to minimize $p_L L + p_K K$ subject to the constraint $U(L, K) = 2$; the firm wants to choose L and K to minimize $p_L L + p_K K$ subject to the constraint $f(L, K) = 2$.

To solve this problem, we need to find the values of our choice variables (L and K) that minimize cost. Since two equations in two unknowns generally yields a unique solution, our approach will be to find two relevant equations involving L and K; solving these simultaneously will give us the answer to the cost minimization problem.

One equation involving L and K is clear from the set-up of the problem. For the individual, we must have $U(L, K) = 2$, i.e., $L^{\frac{1}{2}} K^{\frac{1}{2}} = 2$. For the firm, we must have $f(L, K) = 2$, i.e., $L^{\frac{1}{2}} K^{\frac{1}{2}} = 2$.

Our second constraint is a necessary first-order condition (NFOC) that looks like

$$\frac{\frac{\partial U}{\partial L}}{p_L} = \frac{\frac{\partial U}{\partial K}}{p_K} \quad \text{or} \quad \frac{\frac{\partial f}{\partial L}}{p_L} = \frac{\frac{\partial f}{\partial K}}{p_K}.$$

At the end of this section we will provide three explanations for this NFOC. First, however, we show how to combine the NFOC with the utility (or production) constraint to solve the cost minimization problem.

Solving the Individual's Cost Minimization Problem

Consider an individual with utility function $U = L^{\frac{1}{2}} K^{\frac{1}{2}}$. Assume that the prices of lattes and cake are $p_L = 1$ and $p_K = 2$. What is the minimum cost necessary to reach a utility level of 2?

Well, we know that the solution must satisfy the constraint $L^{\frac{1}{2}}K^{\frac{1}{2}} = 2$, i.e., $LK = 4$. Next, we consider our mysterious NFOC. The partial derivative of utility with respect to L is $\dfrac{\partial U}{\partial L} = \dfrac{1}{2}L^{-\frac{1}{2}}K^{\frac{1}{2}}$; the partial derivative of utility with respect to K is $\dfrac{\partial U}{\partial K} = \dfrac{1}{2}L^{\frac{1}{2}}K^{-\frac{1}{2}}$. Our NFOC is therefore

$$\frac{\frac{\partial U}{\partial L}}{p_L} = \frac{\frac{\partial U}{\partial K}}{p_K} \Longrightarrow \frac{\frac{1}{2}L^{-\frac{1}{2}}K^{\frac{1}{2}}}{1} = \frac{\frac{1}{2}L^{\frac{1}{2}}K^{-\frac{1}{2}}}{2} \Longrightarrow \frac{1}{2}L^{-\frac{1}{2}}K^{\frac{1}{2}} = \frac{1}{4}L^{\frac{1}{2}}K^{-\frac{1}{2}}.$$

Multiplying through by $4L^{\frac{1}{2}}K^{\frac{1}{2}}$ we get

$$2K = L.$$

In other words, the cost-minimizing solution is a consumption bundle with twice as many lattes as pieces of cake.

We can now combine our two equations to find the answer. We know (from the NFOC) that $2K = L$ and (from the utility constraint) that $LK = 4$. Solving simultaneously we get $(2K)K = 4 \Longrightarrow 2K^2 = 4 \Longrightarrow K = \sqrt{2}$. It follows from either of our two equations that the optimal choice of lattes is $L = 2\sqrt{2}$. So the cost minimizing consumption bundle that achieves a utility level of 2 is $(L, K) = (2\sqrt{2}, \sqrt{2})$, and the minimum cost necessary to reach that utility level is

$$C(L, K) = p_L L + p_K K = (1)2\sqrt{2} + (2)\sqrt{2} = 4\sqrt{2}.$$

Solving the Firm's Cost Minimization Problem

Now consider a firm with production function $Y = L^{\frac{1}{2}}K^{\frac{1}{2}}$. The prices of capital and labor are $p_L = 1$ and $p_K = 2$. What is the minimum cost necessary to produce q units of output?

Well, we know that the solution must satisfy the constraint $L^{\frac{1}{2}}K^{\frac{1}{2}} = q$, i.e., $LK = q^2$. Next, we consider our mysterious NFOC. The partial derivative of the production function with respect to L is $\dfrac{\partial f}{\partial L} = \dfrac{1}{2}L^{-\frac{1}{2}}K^{\frac{1}{2}}$; the partial derivative of the production function with respect to K is $\dfrac{\partial f}{\partial K} = \dfrac{1}{2}L^{\frac{1}{2}}K^{-\frac{1}{2}}$. Our NFOC is therefore

$$\frac{\frac{\partial f}{\partial L}}{p_L} = \frac{\frac{\partial f}{\partial K}}{p_K} \Longrightarrow \frac{\frac{1}{2}L^{-\frac{1}{2}}K^{\frac{1}{2}}}{1} = \frac{\frac{1}{2}L^{\frac{1}{2}}K^{-\frac{1}{2}}}{2} \Longrightarrow \frac{1}{2}L^{-\frac{1}{2}}K^{\frac{1}{2}} = \frac{1}{4}L^{\frac{1}{2}}K^{-\frac{1}{2}}.$$

Multiplying through by $4L^{\frac{1}{2}}K^{\frac{1}{2}}$ we get

$$2K = L.$$

In other words, the cost-minimizing solution is an input mix with twice as many units of labor as capital.

We can now combine our two equations to find the answer. We know (from the NFOC) that $2K = L$ and (from the utility constraint) that $LK = q^2$. Solving simultaneously we get $(2K)K = q^2 \Longrightarrow 2K^2 = q^2 \Longrightarrow K = \frac{q}{\sqrt{2}}$. It follows from either of our two equations that the optimal choice of labor is $L = q\sqrt{2}$. So the cost minimizing consumption bundle that achieves an output level of q is $(L, K) = \left(q\sqrt{2}, \frac{q}{\sqrt{2}}\right)$, and the minimum cost necessary to reach that output level is

$$C(L, K) = p_L L + p_K K \Longrightarrow C(q) = (1)q\sqrt{2} + 2\frac{q}{\sqrt{2}} = 2q\sqrt{2}.$$

The function $C(q)$ is the firm's **cost function**: specify how much output you want the firm to produce and $C(q)$ tells you the minimum cost necessary to produce that amount of output. Note that we have transformed the cost function from one involving L and K to one involving q; this will prove useful in deriving supply curves.

Now that we've seen how to use the mysterious NFOC, let's see why it makes sense. The remainder of this section provides three explanations—one intuitive, one graphical, and one mathematical—for our NFOC,

$$\frac{\frac{\partial U}{\partial L}}{p_L} = \frac{\frac{\partial U}{\partial K}}{p_K} \quad \text{or} \quad \frac{\frac{\partial f}{\partial L}}{p_L} = \frac{\frac{\partial f}{\partial K}}{p_K}.$$

An Intuitive Explanation for the NFOC

The first explanation is an intuitive idea called the **last dollar rule**. If our cost-minimizing individual is really minimizing costs, shifting one dollar of spending from cake to lattes cannot increase the individual's utility level; similarly, shifting one dollar of spending from lattes to cake cannot increase the individual's utility level. The individual should therefore be indifferent between spending his "last dollar" on lattes or on cake.

To translate this into mathematics, consider shifting one dollar of spending from *cake* to *lattes*. Such a shift would allow the individual to spend one more dollar on lattes, i.e., to buy $\frac{1}{p_L}$ more lattes; this would increase his utility by $\frac{\partial U}{\partial L} \cdot \frac{1}{p_L}$. (Recall that $\frac{\partial U}{\partial L}$ is the marginal utility of lattes.) But this shift would require him to spend one less dollar on cake, i.e., to buy $\frac{1}{p_K}$ fewer pieces of cake; this would reduce his utility by $\frac{\partial U}{\partial K} \cdot \frac{1}{p_K}$. (Recall that $\frac{\partial U}{\partial K}$ is the marginal utility of cake.) Taken as a whole, this shift cannot increase the individual's utility level, so we must have

$$\frac{\partial U}{\partial L} \cdot \frac{1}{p_L} - \frac{\partial U}{\partial K} \cdot \frac{1}{p_K} \leq 0 \quad \Longrightarrow \quad \frac{\partial U}{\partial L}\frac{1}{p_L} \leq \frac{\partial U}{\partial K}\frac{1}{p_K}.$$

Now consider shifting one dollar of spending from *lattes* to *cake*. Such a shift would allow the individual to spend one more dollar on cake, i.e., to buy $\frac{1}{p_K}$ more pieces of cake; this would increase his utility by $\frac{\partial U}{\partial K}\frac{1}{p_K}$. But this shift would require him to spend one less dollar on lattes, i.e., to buy $\frac{1}{p_L}$ fewer lattes; this would reduce his utility by $\frac{\partial U}{\partial L}\frac{1}{p_L}$. Overall, this shift cannot increase the individual's utility level, so we must have

$$\frac{\partial U}{\partial K} \cdot \frac{1}{p_K} - \frac{\partial U}{\partial L} \cdot \frac{1}{p_L} \le 0 \implies \frac{\partial U}{\partial K}\frac{1}{p_K} \le \frac{\partial U}{\partial L}\frac{1}{p_L}.$$

Looking at the last two equations, we see that

$$\frac{\partial U}{\partial L} \cdot \frac{1}{p_L} \le \frac{\partial U}{\partial K} \cdot \frac{1}{p_K} \qquad \text{and} \qquad \frac{\partial U}{\partial K} \cdot \frac{1}{p_K} \le \frac{\partial U}{\partial L} \cdot \frac{1}{p_L}.$$

The only way both of these equations can hold is if

$$\frac{\partial U}{\partial L} \cdot \frac{1}{p_L} = \frac{\partial U}{\partial K} \cdot \frac{1}{p_K} \qquad \text{i.e.,} \qquad \frac{\frac{\partial U}{\partial L}}{p_L} = \frac{\frac{\partial U}{\partial K}}{p_K}.$$

So if the individual is minimizing cost, this equation must hold.

The identical logic works for firms. If the firm is minimizing costs, shifting one dollar of spending from capital to labor cannot increase the firm's output; similarly, shifting one dollar of spending from labor to capital cannot increase the firm's output. The firm should therefore be indifferent between spending its "last dollar" on labor or on capital.

Mathematically, we end up with

$$\frac{\frac{\partial f}{\partial L}}{p_L} = \frac{\frac{\partial f}{\partial K}}{p_K}.$$

With one extra dollar, the firm could hire $\frac{1}{p_L}$ extra units of labor; the extra output the firm could produce is therefore $\frac{\partial f}{\partial L} \cdot \frac{1}{p_L}$. (Recall that $\frac{\partial f}{\partial L}$ is the marginal product of labor, i.e., the extra output the firm could produce with one extra unit of labor.) Similarly, spending an extra dollar on capital would allow the firm to hire $\frac{1}{p_K}$ extra units of capital; the extra output the firm could produce is therefore $\frac{\partial f}{\partial K} \cdot \frac{1}{p_K}$. (Recall that $\frac{\partial f}{\partial K}$ is the marginal product of capital, i.e., the extra output the firm could produce with one extra unit of capital.) If the firm is minimizing cost, it must be equating these two fractions.

A Graphical Explanation for the NFOC

The second explanation for the NFOC is graphical. Recall from Part I that an individual's **budget constraint** is the set of all consumption bundles (L, K) that an individual can purchase with a given budget. The line $p_L L + p_K K = 10$ is the budget constraint corresponding to a budget of \$10; we can rewrite this as $K = \frac{10 - p_L L}{p_K}$. The slope of the budget constraint, $\frac{dK}{dL} = -\frac{p_L}{p_K}$, measures the **marginal rate of transformation** between lattes and cake. In order to afford an extra latte, the individual needs to give up $\frac{p_L}{p_K}$ pieces of cake in order to stay within his budget. (For example, if lattes cost \$1 and cake costs \$.50 per piece, he would have to give up 2 pieces of cake to afford one extra latte.)

Firms have structures analogous to budget constraints called **isocosts**: the set of all input bundles (L, K) that the firm can purchase with a given budget. The line $p_L L + p_K K = 10$ is the isocost corresponding to a budget of \$10; we can rewrite this as $K = \frac{10 - p_L L}{p_K}$. The slope of the isocost, $\frac{dK}{dL} = -\frac{p_L}{p_K}$, measures the **marginal rate of technical transformation** between labor and capital. In order to afford an extra unit of labor, the firm needs to give up $\frac{p_L}{p_K}$ units of capital in order to stay within its budget. (For example, if labor costs \$1 per unit and capital costs \$.50 per unit, the firm would have to give up 2 units of capital to afford one extra unit of labor.)

Graphically, the cost minimization problem is for the individual to find the lowest budget constraint that intersects a specified indifference curve (or, equivalently, for the firm to find the lowest isocost that intersects a specified isoquant). We can see from Figure 1 that the solution occurs at a point where the budget constraint is tangent to the indifference curve (or, equivalently, where the isocost is tangent to the isoquant). At this point of tangency, the slope of the indifference curve must equal the slope of the budget constraint:

$$-\frac{\frac{\partial U}{\partial L}}{\frac{\partial U}{\partial K}} = -\frac{p_L}{p_K} \implies \frac{\frac{\partial U}{\partial L}}{p_L} = \frac{\frac{\partial U}{\partial K}}{p_K}.$$

Equivalently, in the case of firms we have that the slope of the isoquant must equal the slope of the isocost:

$$-\frac{\frac{\partial f}{\partial L}}{\frac{\partial f}{\partial K}} = -\frac{p_L}{p_K} \implies \frac{\frac{\partial f}{\partial L}}{p_L} = \frac{\frac{\partial f}{\partial K}}{p_K}.$$

A Mathematical Explanation for the NFOC

The third and final explanation for the NFOC comes from brute force mathematics. The individual's problem is to choose L and K to minimize costs $p_L L + p_K K$ subject to a utility constraint $U(L, K) = \overline{U}$. It turns out that we can solve this problem by writing down the **Lagrangian**

$$\mathscr{L} = p_L L + p_K K + \lambda[\overline{U} - U(L, K)].$$

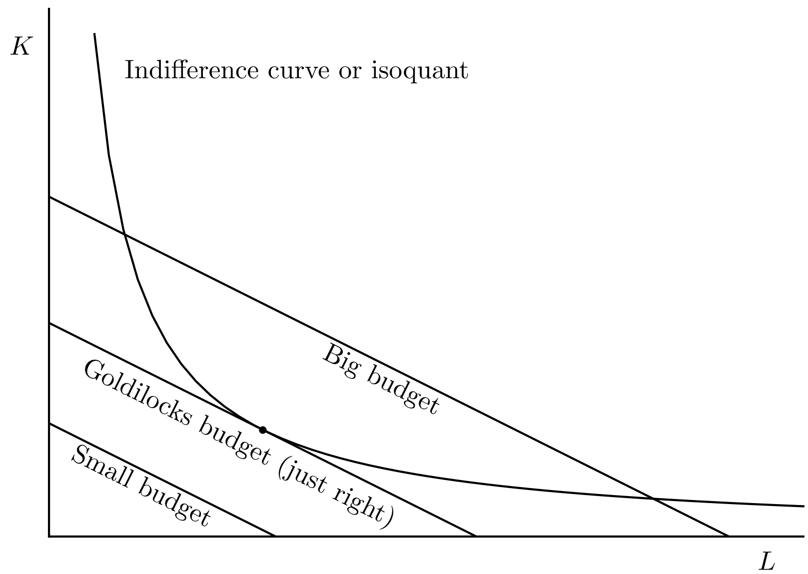

Figure 1: Minimizing costs subject to a utility (or output) constraint

(The Greek letter λ—pronounced "lambda"—is called the Lagrange multiplier; it has important economic meanings that you can learn more about in upper-level classes.) Magically, the necessary first-order conditions (NFOCs) for the individual turn out to be

$$\frac{\partial \mathscr{L}}{\partial L} = 0, \quad \frac{\partial \mathscr{L}}{\partial K} = 0, \quad \text{and } \frac{\partial \mathscr{L}}{\partial \lambda} = 0,$$

i.e.,

$$p_L - \lambda \frac{\partial U}{\partial L} = 0, \quad p_K - \lambda \frac{\partial U}{\partial K} = 0, \quad \text{and } \overline{U} - U(K, L) = 0.$$

Solving the first two for λ we get

$$\lambda = \frac{p_L}{\frac{\partial U}{\partial L}} \quad \text{and} \quad \lambda = \frac{p_K}{\frac{\partial U}{\partial K}}.$$

Setting these equal to each other and rearranging yields the mysterious NFOC,

$$\frac{\frac{\partial U}{\partial L}}{p_L} = \frac{\frac{\partial U}{\partial K}}{p_K}.$$

The mathematics of the firm's problem is identical: choose L and K to minimize costs $p_L L + p_K K$ subject to a production constraint $f(L, K) = \overline{Y}$. The Lagrangian is

$$\mathscr{L} = p_L L + p_K K + \lambda[\overline{Y} - f(L, K)]$$

and the necessary first-order conditions (NFOCs) are

$$\frac{\partial \mathscr{L}}{\partial L} = 0, \quad \frac{\partial \mathscr{L}}{\partial K} = 0, \quad \text{and } \frac{\partial \mathscr{L}}{\partial \lambda} = 0,$$

i.e.,

$$p_L - \lambda \frac{\partial f}{\partial L} = 0, \quad p_K - \lambda \frac{\partial f}{\partial K} = 0, \quad \text{and } \overline{Y} - f(L, K) = 0.$$

Solving the first two for λ we get

$$\lambda = \frac{p_L}{\frac{\partial f}{\partial L}} \quad \text{and} \quad \lambda = \frac{p_K}{\frac{\partial f}{\partial K}}.$$

Setting these equal to each other and rearranging yields the firm's NFOC,

$$\frac{\frac{\partial f}{\partial L}}{p_L} = \frac{\frac{\partial f}{\partial K}}{p_K}.$$

2 Supply Curves

The firm's ultimate job is not to minimize costs but to maximize profits. An examination of profit maximization allows us to derive **supply curves**, which show how a change in the price of the firm's output (p) affect the firm's choice of output (q), *holding other prices constant.* We will also be able to derive the firm's **factor demand curves**, which show how a change in the price of one of the firm's inputs (e.g., p_L, the price of labor) affects the firm's choice of how many units of that input to purchase, *holding other prices constant.*

The caveat "holding other prices constant" arises because of graphical and mental limitations. Supply and demand graphs are only two-dimensional, so we cannot use them to examine how multiple price changes affect the firm's output or input choices; this would require three- or more-dimensional graphs. In fact, what is really going on is this: the firm's profit-maximizing level of output is a function of *all* of the input and output prices. If there are 5 inputs and 1 output, there are 6 prices to consider. Seeing how the profit-maximizing level of output changes as these prices change would require a *seven*-dimensional graph (six dimensions for the prices, and one dimension for the level of output). Unfortunately, we have a hard time visualizing seven-dimensional graphs. To solve this problem, we look at two-dimensional *cross-sections* of this seven-dimensional graph. By changing the output price while holding all of the input prices constant, we are able to see (in two dimensions) how the output price affects the firm's choice of output; in other words, we are able to see the firm's supply curve. Similarly, by changing the price of labor while holding all other prices constant, we are able to see how the price of labor affects the firm's profit-maximizing choice of labor; this gives us the firm's factor demand curve for labor.

An important implication here is that the levels at which we hold other prices constant is important: the firm's supply curve with input prices of $p_L = p_K = 2$ will be different than its supply curve with input prices of $p_L = p_K = 4$.

Choosing Output to Maximize Profits

To derive the firm's supply curve, we hold the input prices constant at specified levels. We can therefore use the firm's cost function $C(q)$ (derived previously) to express the profit maximization problem as the problem of choosing output q to maximize profits,

$$\pi(q) = pq - C(q).$$

To maximize this we take a derivative with respect to our choice variable and set it equal to zero. We get an NFOC of

$$\frac{d\pi}{dq} = 0 \quad \Longrightarrow \quad p - C'(q) = 0 \Longrightarrow p = C'(q).$$

The right hand side here is the marginal cost of producing an additional unit of output. The left hand side is the marginal benefit of producing an additional unit of output, namely the market price p. The NFOC therefore says that a profit-maximizing firm produces until the marginal cost of production is equal to the output price; this is true as long as the firm's profit-maximization problem has an interior solution. It follows that **the marginal cost curve *is* the supply curve** (as long as the firm's profit-maximization problem has an interior solution[2]).

Choosing Inputs to Maximize Profits

To get the factor demand curves, we express the firm's objective in terms of its input choices: it wants to choose inputs L and K to maximize profits

$$\pi(L, K) = pf(L, K) - p_L L - p_K K.$$

To maximize this we take partial derivatives with respect to our choice variables and set them equal to zero. For labor we get an NFOC of

$$\frac{\partial \pi}{\partial L} = 0 \quad \Longrightarrow \quad p\frac{\partial f}{\partial L} - p_L = 0 \quad \Longrightarrow \quad p\frac{\partial f}{\partial L} = p_L.$$

The right hand side here is the marginal cost of purchasing an additional unit of labor, namely its market price p_L. The left hand side is the marginal benefit of purchasing an additional unit of labor, namely the market price of the additional output that can be produced with that extra labor: p is the market price of output, and $\frac{\partial f}{\partial L}$ is the marginal product of labor, i.e., the extra output that can be produced with one more unit of labor. Multiplying these yields the **value of the marginal product of labor** (VMP_L). The NFOC therefore says that a profit-maximizing firm purchases labor until the value of the marginal product of labor VMP_L is equal to the cost of labor p_L. If $VMP_L > p_L$ then the firm

[2] At low prices, the firm's optimal choice of output may be the corner solution $q = 0$. In these cases the marginal cost curve and the supply curve will not be the same.

can increase profits by purchasing additional units of labor; if $VMP_L < p_L$ then the firm can increase profits by purchasing fewer units of labor.

This NFOC turns out to yield the firm's factor demand curve for labor. Given a fixed input of capital and a fixed output price p, the NFOC tells us how much labor the firm wants to hire as a function of the price of labor p_L.

An identical analysis shows that a profit-maximizing firm must purchase capital until the value of the marginal product of capital VMP_K is equal to the cost of capital p_K:

$$\frac{\partial \pi}{\partial K} = 0 \Longrightarrow p\frac{\partial f}{\partial K} - p_K = 0 \Longrightarrow p\frac{\partial f}{\partial K} = p_K.$$

This NFOC gives us the firm's factor demand curve for capital.

3 Demand Curves

The individual's ultimate job is not to minimize costs subject to a utility constraint but to maximize utility subject to a budget constraint. An examination of utility maximization allows us to derive demand curves, which show how a change in the price of some good (e.g., p_L, the price of lattes) affects the individual's consumption choice for that good, *holding other prices constant and holding the individual's budget constant.* We can also derive the individual's **Engel curves**, which show how a change in the individual's *budget* affects the individual's consumption choices for, say, lattes, *holding all prices constant.*

As in the case of firms, the caveat about holding things constant arises because our graphs (and our minds) work best in only two dimensions. If the individual's utility level depends on five different consumption goods and his budget, his optimal choice of lattes (L) would be a seven-dimensional graph: five dimensions for the prices of the different goods, one dimension for the budget, and one dimension for his optimal choice of L. Since we can't visualize seven dimensions, we hold five of these dimensions constant and get a two-dimensional cross-section. Holding the budget and the other four prices constant allows us to see how the price of lattes affects the individual's optimal choice of lattes; this gives us the individual's demand curve for lattes. Holding all of the prices constant allows us to see how the individual's budget affects his choice of lattes; this give us the individual's Engel curves for lattes. As in the case of firms, the levels at which we hold various things constant is important: the individual's demand for lattes with a budget of \$20 and a price of cake of \$1 per piece will be different than his demand for lattes with a budget of \$30 and a price of cake of \$2 per piece.

Marshallian (Money Held Constant) Demand Curves

Consider the problem of choosing consumption amounts L and K to maximize utility $U(L, K)$ subject to a budget constraint $p_L L + p_K K = \overline{C}$. Graphically,

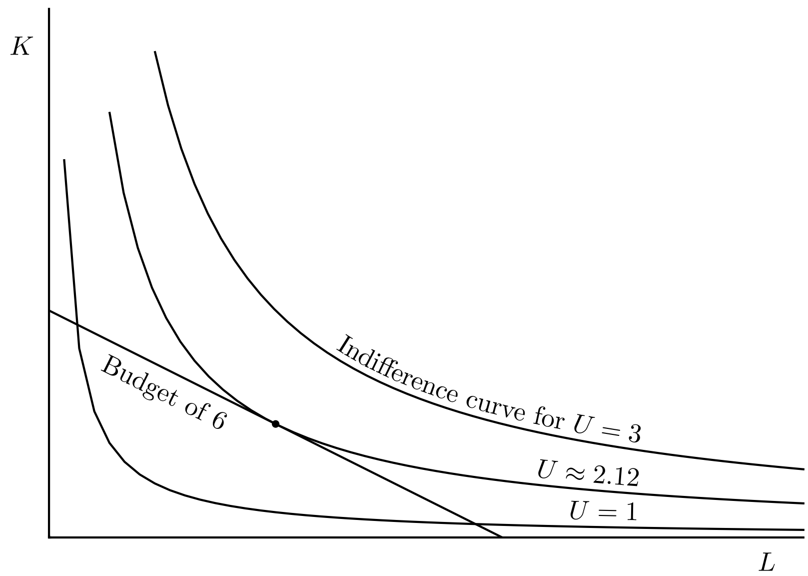

Figure 2: Maximizing utility subject to a budget constraint of 6

this problem (with $p_L = 1, p_K = 2$, and $\overline{C} = 6$) and its solution are shown in Figure 2.

As with cost minimization, our solution method will be to find two equations involving L and K and then solve them simultaneously. The first equation comes from the statement of the problem: L and K must satisfy the budget constraint, $p_L L + p_K K = \overline{C}$.

The second equation turns out to be the same mysterious NFOC we found in the case of cost minimization:

$$\frac{\frac{\partial U}{\partial L}}{p_L} = \frac{\frac{\partial U}{\partial K}}{p_K}.$$

Intuitively, this is because the utility-maximizing individual must still follow the last dollar rule: if he is choosing optimally, he should be indifferent between spending his last dollar on lattes or on cake. Graphically, we can see from Figure 2 that the individual's task is to choose the highest indifference curve that lies on the relevant budget constraint; the optimum occurs at a point of tangency between the indifference curve and the budget constraint, meaning that the slope of the indifference curve must equal the slope of the budget constraint. Mathematically, we can write down the relevant Lagrangian,

$$\mathscr{L} = U(L, K) - \lambda[\overline{C} - p_L L - p_K K].$$

The NFOCs include $\frac{\partial \mathscr{L}}{\partial L} = 0$ and $\frac{\partial \mathscr{L}}{\partial K} = 0$, i.e., $\frac{\partial U}{\partial L} - \lambda p_L = 0$ and $\frac{\partial U}{\partial K} - \lambda p_K = 0$. Solving these for λ and setting them equal to each other yields the desired

NFOC. Combining the budget constraint with the NFOC allow us to solve for the individual's **Marshallian demand curves**, Engel curves, &etc.

For example, consider an individual with a budget of \$6 and a utility function of $U = L^{\frac{1}{2}}K^{\frac{1}{2}}$. The price of cake is $p_K = 2$. What is this individual's Marshallian demand curve for lattes?

To solve this problem, we substitute the budget ($\overline{C} = \$6$) and the price of cake (\$2 per piece) into the budget constraint to get $p_L L + 2K = 6$. Next, we substitute the price of cake into the NFOC to get

$$\frac{\frac{\partial U}{\partial L}}{p_L} = \frac{\frac{\partial U}{\partial K}}{p_K} \Longrightarrow \frac{\frac{1}{2}L^{-\frac{1}{2}}K^{\frac{1}{2}}}{p_L} = \frac{\frac{1}{2}L^{\frac{1}{2}}K^{-\frac{1}{2}}}{2} \Longrightarrow \frac{1}{2p_L}L^{-\frac{1}{2}}K^{\frac{1}{2}} = \frac{1}{4}L^{\frac{1}{2}}K^{-\frac{1}{2}}.$$

Multiplying through by $4p_L L^{\frac{1}{2}}K^{\frac{1}{2}}$ we get

$$2K = p_L L.$$

We now have two equations ($p_L L + 2K = 6$ and $2K = p_L L$) in three unknowns (L, K, and p_L). We can eliminate K by using the second equation to substitute for $2K$ in the first equation: $p_L L + p_L L = 6$. This simplifies to $2p_L L = 6$ or $p_L L = 3$, which we can rewrite as

$$L = \frac{3}{p_L}.$$

This is the Marshallian demand curve for lattes when we fix the budget at 6 and the price of cake at \$2 per piece. If the price of lattes is $p_L = 1$, the optimal choice is $L = 3$; if the price of lattes is $p_L = 3$, the optimal choice is $L = 1$.

Hicksian (Utility Held Constant) Demand Curves

For reasons to be discussed in the next section, it turns out that we should also consider the demand curves that come from the cost minimization problem discussed previously: choose consumption amounts L and K to minimize costs $C(L, K) = p_L L + p_K K$ subject to a utility constraint $U(L, K) = \overline{U}$. Graphically, this problem (with $p_L = 1, p_K = 2$, and $\overline{U} = 2$) and its solution are shown in Figure 3.

To solve this problem, we combine our utility constraint, $U(L, K) = \overline{U}$, with the usual NFOC,

$$\frac{\frac{\partial U}{\partial L}}{p_L} = \frac{\frac{\partial U}{\partial K}}{p_K}.$$

Combining the budget constraint with the NFOC allow us to solve for the individual's **Hicksian demand curves**.

For example, consider an individual with a utility constraint of $\overline{U} = 2$ and a utility function of $U = L^{\frac{1}{2}}K^{\frac{1}{2}}$. The price of cake is $p_K = 2$. What is this individual's Hicksian demand curve for lattes?

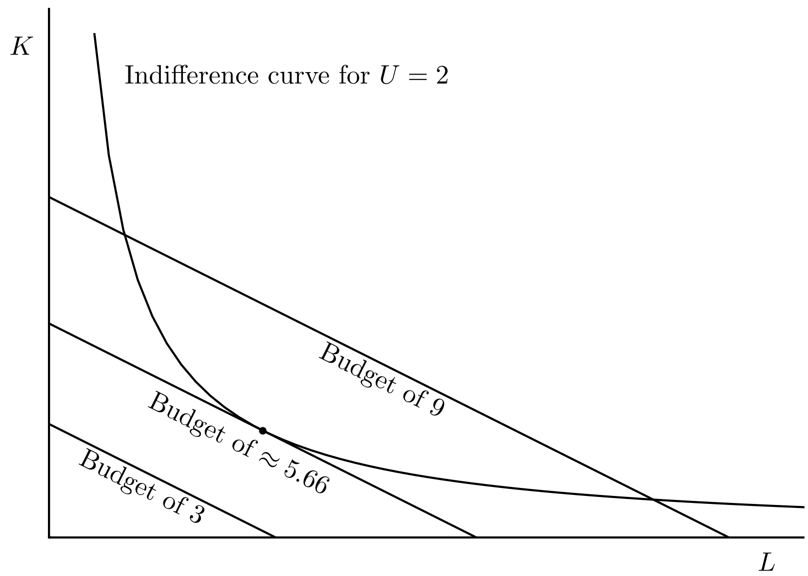

Figure 3: Minimizing costs subject to a utility constraint

To solve this problem, we substitute the utility level ($\overline{U} = 2$) into the utility constraint to get $L^{\frac{1}{2}}K^{\frac{1}{2}} = 2$. Next, we substitute the price of cake (\$2 per piece) into the NFOC to get

$$\frac{\frac{\partial U}{\partial L}}{p_L} = \frac{\frac{\partial U}{\partial K}}{p_K} \Longrightarrow \frac{\frac{1}{2}L^{-\frac{1}{2}}K^{\frac{1}{2}}}{p_L} = \frac{\frac{1}{2}L^{\frac{1}{2}}K^{-\frac{1}{2}}}{2} \Longrightarrow \frac{1}{2p_L}L^{-\frac{1}{2}}K^{\frac{1}{2}} = \frac{1}{4}L^{\frac{1}{2}}K^{-\frac{1}{2}}.$$

Multiplying through by $4p_L L^{\frac{1}{2}}K^{\frac{1}{2}}$ we get

$$2K = p_L L.$$

We now have two equations ($L^{\frac{1}{2}}K^{\frac{1}{2}} = 2$ and $2K = p_L L$) in three unknowns (L, K, and p_L). We can eliminate K by using the second equation to substitute for K in the first equation:

$$L^{\frac{1}{2}}\left(\frac{p_L L}{2}\right)^{\frac{1}{2}} = 2 \Longrightarrow L\left(\frac{p_L}{2}\right)^{\frac{1}{2}} = 2 \Longrightarrow L\sqrt{p_L} = 2\sqrt{2}.$$

We can rewrite this as

$$L = \frac{2\sqrt{2}}{\sqrt{p_L}}$$

This is the Hicksian demand curve for lattes when we fix utility at $U = 2$ and the price of cake at \$2 per piece. If the price of lattes is $p_L = 1$, the optimal choice is $L = 2\sqrt{2}$; if the price of lattes is $p_L = 2$, the optimal choice is $L = 2$.

Marshall v. Hicks

Why do we use both Marshallian and Hicksian demand curves? Well, the advantage of the Marshallian demand curve is that it matches real life: people actually do have budget constraints, and they have to do the best they can subject to those constraints. The advantage of the Hicksian demand curve is that it makes for nice economic theory. For example, we can prove that Hicksian demand curves are downward sloping: people buy less as prices rise. And the area underneath a Hicksian demand curve represents a useful concept called **consumer surplus**. Unfortunately, Marshallian demand curves do not have nice theoretical properties; Marshallian demand curves do not have to be downward sloping, and the area underneath a Marshallian demand curve *does not* represent consumer surplus (or anything else interesting).

We are therefore in the uncomfortable situation of getting pulled in two different directions: reality pulls us toward Marshallian demand curves and theory pulls us toward Hicksian demand curves. There is, however, a silver lining in this cloud: comparing Marshallian and Hicksian demand curves provides insight into both of them.

Consider what happens if, say, the price of lattes increases. In the Marshallian approach, your purchasing power declines, and you will end up on a lower indifference curve (i.e., with a lower utility level). In the Hicksian approach, your budget adjusts upward to *compensate* you for the increased price of lattes, and you end up on the same indifference curve (i.e., with the same utility level). For this reason the Hicksian demand curve is also called the **compensated demand curve**, while the Marshallian demand curve is called the **uncompensated demand curve**.[3]

An obvious question here is: What exactly is the Hicksian (or compensated) demand curve compensating you *for*? This answer is, *for changes in the purchasing power of your budget.* The Hicksian approach increases or decreases your income so that you're always on the same indifference curve, regardless of price changes. The Hicksian demand curve therefore represents a pure **substitution effect**: when the price of lattes increases in the Hicksian world, you buy fewer lattes—not because your budget is smaller, but because lattes now look more expensive compared to cake. This change in *relative* prices induces you to buy fewer lattes and more cake, i.e., to substitute out of lattes and into cake.

The substitution effect is also present in the Marshallian approach. When the price of lattes increases, you buy fewer lattes because of the change in relative prices. But there is an additional factor: the increased price of lattes reduces your *purchasing power*. This is called an **income effect** because a loss of purchasing power is essentially a loss of income. Similarly, a decrease in the price of lattes increases your purchasing power; the income effect here is essentially a gain in income.

[3] Of course, the Hicksian compensation scheme can work against you, too. If the price of lattes *decreases* then the Marshallian approach puts you on a higher indifference curve, i.e., with a higher utility level. The Hicksian approach reduces your budget so that you end up on the same indifference curve, i.e., with the same utility level.

The Marshallian demand curve therefore incorporates both income and substitution effects; the Hicksian demand curve only incorporates the substitution effect. If you take more economics, you can prove this mathematically using the Slutsky equation, which relates these two types of demand curves.

For our purposes, it is more important to do a qualitative comparison. We can prove that Hicksian demand curves are downward sloping (i.e., that people buy fewer lattes when the price of lattes goes up) because the substitution effect always acts in the *opposite* direction of the price change: when prices go up the substitution effect induces people to buy less, and when prices go down the substitution effect induces people to buy more.

The income effect is more difficult. For **normal goods**, an increase in income leads you to buy more: your purchasing power rises, so you buy more vacations and more new computers &etc. But for **inferior goods**, an increase in your income leads you to buy *less*: your purchasing power rises, so you buy less Ramen and less used underwear &etc. In a parallel fashion, a reduction in your income leads you to buy more inferior goods: you now have less money, so you buy *more* Ramen and more used underwear &etc.

In sum, while the substitution effect always acts in the opposite direction of the price change, the income effect can go in either direction: increases or decreases in income can either increase or decrease your optimal consumption of some item, depending on whether the item is a normal good or an inferior good. In some instances (rare in practice but possible in theory), the income effect for an inferior good can overwhelm the substitution effect. Consider, for example, an individual who consumes Ramen (an inferior good) and steak (a normal good). If the price of Ramen goes up, the substitution effect will lead the individual to substitute out of Ramen and into steak, i.e., to consume less Ramen. But the increased price of Ramen will also reduce the individual's purchasing power, and this income effect will lead him to consume *more* Ramen. If the income effect is strong enough, the overall result may be that an increase in the price of Ramen leads the individual to consume *more* Ramen. In other words, we have an *upward sloping* Marshallian demand curve! A good with an upward sloping Marshallian demand curve is called a **Giffen good**.