Problem Set #1: Calculus and Micro Theory

1. Explain intuitively the importance of taking derivatives and setting them equal to zero.

2. Use the definition of a derivative to prove that constants pass through derivatives, i.e., that $\frac{d}{dx}[(c \cdot f(x)] = c \cdot \frac{d}{dx}[f'(x)]$.

3. Use the product rule to prove that the derivative of x^2 is $2x$. (*Challenge*: Do the same for higher-order integer powers, e.g., x^{30}. *Do not* do this the hard way.)

4. For each of the following functions, calculate the first derivative, the second derivative, and determine maximum and/or minimum values (if they exist):

   (a) $x^2 + 2$

   (b) $(x^2 + 2)^2$

   (c) $(x^2 + 2)^{\frac{1}{2}}$

   (d) $-x(x^2 + 2)^{\frac{1}{2}}$

   (e) $\ln\left[(x^2 + 2)^{\frac{1}{2}}\right]$

5. Calculate partial derivatives with respect to x and y of the following functions:

   (a) $x^2y - 3x + 2y$

   (b) e^{xy}

   (c) $e^x y^2 - 2y$

6. Imagine that a monopolist is considering entering a market with demand curve $q = 20 - p$. Building a factory will cost F, and producing each unit will cost 2 so its profit function (if it decides to enter) is $\pi = pq - 2q - F$.

   (a) Substitute for p using the inverse demand curve and find the (interior) profit-maximizing level of output for the monopolist. Find the profit-maximizing price and the profit-maximizing profit level.

   (b) For what values of F will the monopolist choose not to enter the market?

7. (Profit maximization for a firm in a competitive market) Profit is $\pi = p \cdot q - C(q)$. If the firm is maximizing profits and takes p as given, find the necessary first order condition for an interior solution to this problem, both in general and in the case where $C(q) = \dfrac{1}{2}q^2 + 2q$.

8. (Profit maximization for a non-price-discriminating monopolist) A monopolist can choose both price and quantity, but choosing one essentially determines the other because of the constraint of the market demand curve: if you choose price, the market demand curve tells you how many units you can sell at that price; if you choose quantity, the market demand curve tells you the maximum price you can charge while still selling everything you produce. So: if the monopolist is profit-maximizing, find the necessary first order condition for an interior solution to the monopolist's problem, both in general and in the case where the demand curve is $q = 20 - p$ and the monopolist's costs are $C(q) = \frac{1}{2}q^2 + 2q$.

9. Use intuition, graphs, or math to explain the "mysterious NFOC" (a.k.a. the last dollar rule).

10. Consider the production function $f(L, K) = L^{\frac{1}{4}}K^{\frac{1}{2}}$.

    (a) What is the equation for the isoquant corresponding to an output level of q?

    (b) What is the slope $\frac{dK}{dL}$ of the isoquant, i.e., the marginal rate of technical substitution?

    (c) Explain intuitively what the marginal rate of technical substitution measures.

    (d) Assume that the prices of L and K are $p_L = 2$ and $p_K = 2$. Write down the problem for minimizing cost subject to the constraint that output must equal q. Clearly specify the objective function and the choice variables.

    (e) Explain how to go about solving this problem.

    (f) Solve this problem, i.e., find the minimum cost $C(q)$ required to reach an output level of q. What are the optimal choices of L and K? (Note: these will be functions of q. You may wish to do the next problem first if you're getting confused by all the variables.)

    (g) What is the minimum cost required to reach an output level of 10? What are the optimal choices of L and K?

11. Consider the production function $f(L, K) = L^{\frac{1}{4}}K^{\frac{1}{2}}$. Assume (as above) that the firm's input prices are $p_L = p_K = 2$; also assume that the price of the firm's output is p.

    (a) Write down the problem of choosing output q to maximize profits. Use the cost function $C(q) \approx 3.78q^{\frac{4}{3}}$ (which you derived above) to represent costs. Clearly specify the objective function and the choice variables.

    (b) Explain how to go about solving this problem. Also explain why it's kosher to substitute $C(q)$ into the profit function, i.e., explain why cost-minimization is a necessary condition for profit-maximization.

(c) Solve this problem to derive the firm's supply curve. Use approximations where helpful.

(d) If the price of output is $p = 16$, how much will the firm produce? What will its profits be?

12. The previous problems have dealt with **long run** cost curves and supply curves, meaning that the firm has complete control over all of its inputs. In the **short run**, however, the firm cannot change its capital stock—it can choose how much labor to hire, but it can't build any new factories. In this problem we will examine short run cost curves and short run supply curves.

    (a) Assume that the firm's production function is $f(L, K) = L^{\frac{1}{4}} K^{\frac{1}{2}}$, and that capital is fixed at $K = 4$. What is the equation for the isoquant corresponding to an output level of q?

    (b) Assume further that the prices of L and K are $p_L = 2$ and $p_K = 2$. Write down the problem for minimizing cost subject to the constraint that output must equal q. Clearly specify the objective function and the choice variables.

    (c) In a previous problem you provided an intuitive explanation for the marginal rate of technical substitution. Given that capital is fixed at $K = 4$, what is the relevance (if any) of this concept in the present problem?

    (d) How will the price of capital p_K affect the firm's behavior?

    (e) Solve this problem, i.e., find the minimum cost $C(q)$ required to reach an output level of q. What is the optimal choice of L?

    (f) Write down the profit maximization problem, using the function $C(q)$ you found above. Calculate the firm's short run supply curve.

13. Consider a firm with production function $f(L, K) = L^{\frac{1}{2}} K^{\frac{1}{2}}$ and input prices of $p_L = 1$ and $p_K = 2$.

    (a) Calculate the supply curve for this firm. (Note: the related cost-minimization problem was done in the text, with an answer of $C(q) = 2q\sqrt{2}$.)

    (b) How much will the firm supply at a price of $p = 2$? *Hint: Think about corner solutions!*

    (c) How much will the firm supply at a price of $p = 4$?

    (d) Show that this firm's production function exhibits **constant returns to scale**, i.e., that doubling inputs doubles output, i.e., $f(2L, 2K) = 2f(L, K)$.

    (e) Does the idea of constant returns to scale help explain the firm's behavior when $p = 4$? *Hint: Think about this in the context of the objective function.* If it does help you explain the firm's behavior, you may find value in knowing that **increasing returns to scale** occurs when doubling inputs more than doubles outputs, i.e., $f(2L, 2K) > 2f(L, K)$, and that **decreasing returns to scale** occurs when doubling inputs less than doubles outputs, i.e., $f(2L, 2K) < 2f(L, K)$. An industry with constant or increasing returns to scale can often lead to monopolization of production by a single company.