Answer Key to Problem Set #1: Calculus and Micro Theory

1. Explain the importance of taking derivatives and setting them equal to zero.

   Microeconomics is about the actions and interactions of optimizing agents (e.g., profit-maximizing firms, utility-maximizing consumers). For differentiable functions with interior maxima or minima, the way to find those interior maxima or minima is to take a derivative and set it equal to zero. This gives you *candidate values* for maxima or minima; the reason is that slopes (i.e., derivatives) are equal to zero at the top of a hill (a maximum) or at the bottom of a valley (a minimum).

2. Use the definition of a derivative to prove that constants pass through derivatives, i.e., that $\frac{d}{dx}[(c \cdot f(x)] = c \cdot \frac{d}{dx}[f'(x)]$.

$$\begin{aligned} \frac{d}{dx}[(c \cdot f(x)] &= \lim_{h \to 0} \frac{c \cdot f(x+h) - c \cdot f(x)}{h} \\ &= c \cdot \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = c \cdot \frac{d}{dx}[f'(x)]. \end{aligned}$$

3. Use the product rule to prove that the derivative of x^2 is $2x$. (*Challenge*: Do the same for higher-order integer powers, e.g., x^{30}. *Do not* do this the hard way.)

   $\frac{d}{dx}(x^2) = \frac{d}{dx}(x \cdot x) = x \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(x) = 2x$.

   For higher powers, use induction: We've just proved that our rule ($\frac{d}{dx}(x^n) = nx^{n-1}$) is true for $n = 2$. So now we assume that it's true for n ($\frac{d}{dx}(x^n) = nx^{n-1}$) and need to show that it's true for x^{n+1}. But we can just use the same trick again:

   $\frac{d}{dx}(x^{n+1}) = \frac{d}{dx}(x \cdot x^n) = x \cdot \frac{d}{dx}(x^n) + x^n \cdot \frac{d}{dx}(x) = xnx^{n-1} + x^n = (n+1)x^n$.

4. For each of the following functions, calculate the first derivative, the second derivative, and determine maximum and/or minimum values (if they exist):

   (a) $x^2 + 2$

   We have $f'(x) = 2x$ and $f''(x) = 2$. Candidate solutions for interior maxima or minima are where $f'(x) = 0$. The only candidate is $x = 0$, which turns out to be a minimum value of the function $f(x)$. Note that the sign of the second derivative, $f''(0) = 2 > 0$, identifies this as a minimum; this is also clear from a graph of $f(x)$.

(b) $(x^2+2)^2$

We have $f'(x) = 2(x^2+2)\cdot\frac{d}{dx}(x^2+2) = 4x(x^2+2) = 4x^3+8x$ and $f''(x) = 12x^2+8$. Candidate solutions for interior maxima or minima are where $f'(x)=0$. The only candidate is $x=0$, which turns out to be a minimum value of the function $f(x)$. (We are not interested in imaginary roots such as $i\sqrt{2}$.) Note that the sign of the second derivative, $f''(0)=8>0$, identifies this as a minimum.

(c) $(x^2+2)^{\frac{1}{2}}$

We have $f'(x) = \frac{1}{2}\cdot(x^2+2)^{-\frac{1}{2}}\cdot\frac{d}{dx}(x^2+2) = x\cdot(x^2+2)^{-\frac{1}{2}}$ and

$$\begin{aligned}
f''(x) &= x\cdot\frac{d}{dx}\left[(x^2+2)^{-\frac{1}{2}}\right] + \left[(x^2+2)^{-\frac{1}{2}}\right]\cdot\frac{d}{dx}(x) \\
&= x\cdot\left(-\frac{1}{2}\right)\cdot(x^2+2)^{-\frac{3}{2}}\frac{d}{dx}(x^2+2) + \left[(x^2+2)^{-\frac{1}{2}}\right]\cdot 1 \\
&= -x^2\cdot(x^2+2)^{-\frac{3}{2}} + (x^2+2)^{-\frac{1}{2}}
\end{aligned}$$

Candidate solutions for interior maxima or minima are where $f'(x)=0$. The only candidate is $x=0$, which turns out to be a minimum value of the function $f(x)$. Note that the sign of the second derivative, $f''(0)=\frac{1}{\sqrt{2}}>0$, identifies this as a minimum.

(d) $-x(x^2+2)^{\frac{1}{2}}$

We have

$$\begin{aligned}
f'(x) &= -x\cdot\frac{d}{dx}\left[(x^2+2)^{\frac{1}{2}}\right] + \left[(x^2+2)^{\frac{1}{2}}\right]\cdot\frac{d}{dx}(-x) \\
&= -x\cdot\left(\frac{1}{2}\right)\cdot(x^2+2)^{-\frac{1}{2}}\frac{d}{dx}(x^2+2) + \left[(x^2+2)^{\frac{1}{2}}\right]\cdot(-1) \\
&= -x^2\cdot(x^2+2)^{-\frac{1}{2}} - (x^2+2)^{\frac{1}{2}} \\
\text{and } f''(x) &= \frac{d}{dx}\left[-x^2\cdot(x^2+2)^{-\frac{1}{2}}\right] - \frac{d}{dx}\left[(x^2+2)^{\frac{1}{2}}\right] \\
&= -x^2\cdot\frac{d}{dx}\left[(x^2+2)^{-\frac{1}{2}}\right] + (x^2+2)^{-\frac{1}{2}}\cdot\frac{d}{dx}\left(-x^2\right) \\
&\quad -\frac{1}{2}\left[(x^2+2)^{-\frac{1}{2}}\right]\frac{d}{dx}(x^2+2) \\
&= -x^2\cdot\left(-\frac{1}{2}\right)\left[(x^2+2)^{-\frac{3}{2}}\right]\frac{d}{dx}(x^2+2) - 2x(x^2+2)^{-\frac{1}{2}} \\
&\quad -x(x^2+2)^{-\frac{1}{2}} \\
&= x^3(x^2+2)^{-\frac{3}{2}} - 3x(x^2+2)^{-\frac{1}{2}}
\end{aligned}$$

Candidate solutions for interior maxima or minima are where $f'(x)=0$. Multiplying both sides by $(x^2+2)^{\frac{1}{2}}$ we get $-x^2-(x^2+2)=0$, which simplifies to $x^2=-1$. Since this equation has no solutions, $f(x)$ has no interior maxima or minima.

(e) $\ln\left[(x^2+2)^{\frac{1}{2}}\right]$

We have

$$\begin{aligned} f'(x) &= \frac{1}{(x^2+2)^{\frac{1}{2}}} \cdot \frac{d}{dx}\left[(x^2+2)^{\frac{1}{2}}\right] \\ &= (x^2+2)^{-\frac{1}{2}} \cdot x \cdot (x^2+2)^{-\frac{1}{2}} \text{(from (c))} \\ &= x \cdot (x^2+2)^{-1} \\ \text{and } f''(x) &= x \cdot \frac{d}{dx}\left[(x^2+2)^{-1}\right] + \left[(x^2+2)^{-1}\right]\frac{d}{dx}(x) \\ &= x(-1)(x^2+2)^{-2}\frac{d}{dx}(x^2+2) + (x^2+2)^{-1} \\ &= -2x^2(x^2+2)^{-2} + (x^2+2)^{-1} \end{aligned}$$

Candidate solutions for interior maxima or minima are where $f'(x) = 0$. The only candidate is $x = 0$, which turns out to be a minimum value of the function $f(x)$. Note that the sign of the second derivative, $f''(0) = \frac{1}{2} > 0$, identifies this as a minimum.

5. Calculate partial derivatives with respect to x and y of the following functions:

(a) $x^2y - 3x + 2y$

We have

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial x}(-3x) + \frac{\partial}{\partial x}(2y) \\ &= y \cdot \frac{\partial}{\partial x}(x^2) - 3 \cdot \frac{\partial}{\partial x}(x) + 0 \\ &= 2xy - 3 \\ \text{and } \frac{\partial}{\partial y} &= \frac{\partial}{\partial y}(x^2y) + \frac{\partial}{\partial y}(-3x) + \frac{\partial}{\partial y}(2y) \\ &= x^2 \cdot \frac{\partial}{\partial y}(y) + 0 + 2 \cdot \frac{\partial}{\partial y}(y) \\ &= x^2 + 2 \end{aligned}$$

(b) e^{xy}

We have $\frac{\partial}{\partial x} = e^{xy} \cdot \frac{\partial}{\partial x}(xy) = ye^{xy}$. Since $f(x,y)$ is symmetric in x and y, we must also have $\frac{\partial}{\partial y} = xe^{xy}$.

(c) $e^xy^2 - 2y$

We have $\frac{\partial}{\partial x} = y^2 \cdot \frac{\partial}{\partial x}(e^x) - 0 = y^2e^x$ and $\frac{\partial}{\partial x} = e^x \cdot \frac{\partial}{\partial x}(y^2) - 2 = 2ye^x - 2$.

6. Imagine that a monopolist is considering entering a market with demand curve $q = 20 - p$. Building a factory will cost F, and producing each unit will cost 2 so its profit function (if it decides to enter) is $\pi = pq - 2q - F$.

(a) Substitute for p using the inverse demand curve and find the (interior) profit-maximizing level of output for the monopolist. Find the profit-maximizing price and the profit-maximizing profit level.

The inverse demand curve is $p = 20 - q$, and substituting into the profit function yields $\pi = (20-q)q - 2q - F = 18q - q^2 - F$. Taking a derivative and setting it equal to zero gives us our candidate solution for an interior maximum: $\pi' = 0 \Longrightarrow 18 - 2q = 0 \Longrightarrow q^* = 9$. Substituting this back into the inverse demand curve yields $p^* = 11$, so that the profit-maximizing profit level is $\pi^* = 11 \cdot 9 - 2 \cdot 9 - F = 81 - F$.

(b) For what values of F will the monopolist choose not to enter the market?

We can see from above that the monopolist will choose not to enter for $F > 81$: zero profits are better than negative profits. Note that we would get a **corner solution** to the maximization problem in this case. The answer $q^* = pi^* = 0$ does *not* show up as one of our candidate interior solutions.

7. (Profit maximization for a firm in a competitive market) Profit is $\pi = p \cdot q - C(q)$. If the firm is maximizing profits and takes p as given, find the necessary first order condition for an interior solution to this problem, both in general and in the case where $C(q) = \frac{1}{2}q^2 + 2q$.

In a competitive market the firm is assumed to be so small that its choice of q doesn't affect the market price p. So the firm treats p like a constant and takes a derivative of the profit function to find candidate (interior) maxima: $\pi' = 0 \Longrightarrow p - C'(q) = 0 \Longrightarrow p = C'(q^*)$. This says that the firm should produce until price equals marginal cost. Give $C(q)$ as above, we get $p = q + 2 \Longrightarrow q^* = p - 2$.

8. (Profit maximization for a non-price-discriminating monopolist) A monopolist can choose both price and quantity, but choosing one essentially determines the other because of the constraint of the market demand curve: if you choose price, the market demand curve tells you how many units you can sell at that price; if you choose quantity, the market demand curve tells you the maximum price you can charge while still selling everything you produce. So: if the monopolist is profit-maximizing, find the necessary first order condition for an interior solution to the monopolist's problem, both in general and in the case where the demand curve is $q = 20 - p$ and the monopolist's costs are $C(q) = \frac{1}{2}q^2 + 2q$.

If the monopolist chooses to produce q, the inverse demand curve establishes the maximum price as $p(q)$. Substituting this into the profit function gives $\pi(p, q) = pq - C(q) \Longrightarrow \pi(q) = p(q) \cdot q - C(q)$. Taking a derivative and setting it equal to zero to find candidate (interior) maxima yields $p(q) \cdot 1 + p'(q) \cdot q - C'(q) = 0$. Given the specific demand and cost curves above, we get $(20 - q) + (-1)q - (q + 2) = 0 \Longrightarrow q^* = 6$.

9. Use intuition, graphs, or math to explain the "mysterious NFOC" (a.k.a. the last dollar rule).

   These are explained to the best of my abilities in the handout.

10. Consider the production function $f(L, K) = L^{\frac{1}{4}} K^{\frac{1}{2}}$.

    (a) What is the equation for the isoquant corresponding to an output level of q?

    The isoquant is $f(L, K) = q$, i.e., $L^{\frac{1}{4}} K^{\frac{1}{2}} = q$.

    (b) What is the slope $\frac{dK}{dL}$ of the isoquant, i.e., the marginal rate of technical substitution?

    Squaring both sides yields $L^{\frac{1}{2}} K = q^2$, i.e., $K = q^2 L^{-\frac{1}{2}}$. The slope of this is $\frac{dK}{dL} = -\frac{1}{2} q^2 L^{-\frac{3}{2}}$.

    (c) Explain intuitively what the marginal rate of technical substitution measures.

    The marginal rate of technical substitute tells you how much capital the firm is willing to trade for one more unit of labor. If $MRTS = -3$, the firm should be willing to trade up to 3 units of capital to gain one unit of labor; such a trade would leave the firm on the same isoquant, i.e., allow the firm to produce the same level of output.

    (d) Assume that the prices of L and K are $p_L = 2$ and $p_K = 2$. Write down the problem for minimizing cost subject to the constraint that output must equal q. Clearly specify the objective function and the choice variables.

    The firm wants to choose L and K to minimize $C(L, K) = 2L + 2K$ subject to $f(L, K) = q$. The choice variables are L and K; the objective function is the cost function $C(L, K)$.

    (e) Explain how to go about solving this problem.

    Two equations in two unknowns usually yield a unique solution, so to solve this problem we will find two relevant equations involving L and K and solve them simultaneously. The first equation is the constraint, $f(L, K) = q$. The second equation comes from the last dollar rule: $\frac{\frac{\partial f}{\partial L}}{p_L} = \frac{\frac{\partial f}{\partial K}}{p_K}$.

    (f) Solve this problem, i.e., find the minimum cost $C(q)$ required to reach an output level of q. What are the optimal choices of L and K? (Note: these will be functions of q. You may wish to do the next problem first if you're getting confused by all the variables.)

    The marginal product of labor is $\frac{\partial f}{\partial L} = \frac{1}{4} L^{-\frac{3}{4}} K^{\frac{1}{2}}$ and the marginal product of capital is $\frac{\partial f}{\partial K} = \frac{1}{2} L^{\frac{1}{4}} K^{-\frac{1}{2}}$, so the last dollar rule gives us

$$\frac{\frac{\partial f}{\partial L}}{p_L} = \frac{\frac{\partial f}{\partial K}}{p_K} \implies \frac{\frac{1}{4} L^{-\frac{3}{4}} K^{\frac{1}{2}}}{2} = \frac{\frac{1}{2} L^{\frac{1}{4}} K^{-\frac{1}{2}}}{2},$$

which simplifies as

$$\frac{1}{4}L^{-\frac{3}{4}}K^{\frac{1}{2}} = \frac{1}{2}L^{\frac{1}{4}}K^{-\frac{1}{2}} \Longrightarrow K = 2L.$$

Substituting this into our first equation, the constraint $L^{\frac{1}{4}}K^{\frac{1}{2}} = q$, yields $L^{\frac{1}{4}}(2L)^{\frac{1}{2}} = q$, which simplifies to $L^{\frac{3}{4}}2^{\frac{1}{2}} = q$, and then to $L^{\frac{3}{4}} = q \cdot 2^{-\frac{1}{2}}$, and finally to $L = q^{\frac{4}{3}}2^{-\frac{2}{3}}$. Substituting this value of L into either of our two equations yields $K = q^{\frac{4}{3}}2^{\frac{1}{3}}$. So the minimum cost to produce 10 units of output is $p_L L + p_K K = 2(q^{\frac{4}{3}}2^{-\frac{2}{3}}) + 2(q^{\frac{4}{3}}2^{\frac{1}{3}}) = q^{\frac{4}{3}}(2^{\frac{1}{3}} + 2^{\frac{4}{3}}) \approx 3.78q^{\frac{4}{3}}$.

(g) What is the minimum cost required to reach an output level of 10? What are the optimal choices of L and K?

Plugging in $q = 10$ yields $L \approx 13.57$, $K \approx 27.14$, and a minimum cost of $p_L L + p_K K \approx 2(13.57) + 2(27.14) = 81.42$.

11. Consider the production function $f(L, K) = L^{\frac{1}{4}}K^{\frac{1}{2}}$. Assume (as above) that the firm's input prices are $p_L = p_K = 2$; also assume that the price of the firm's output is p.

(a) Write down the problem of choosing output q to maximize profits. Use the cost function $C(q) \approx 3.78q^{\frac{4}{3}}$ (which you derived above) to represent costs. Clearly specify the objective function and the choice variables.

The firm's job is to choose q to maximize profits $\pi = pq - C(q) = pq - 3.78q^{\frac{4}{3}}$.

(b) Explain how to go about solving this problem. Also explain why it's kosher to substitute $C(q)$ into the profit function, i.e., explain why cost-minimization is a necessary condition for profit-maximization.

Take a derivative with respect to the choice variable (q) and set it equal to zero. Cost minimization is a necessary condition for profit maximization because a firm that is producing q units of output in a non-cost-minimizing way can always increase profits by producing q units of output in the cost-minimizing way. So a profit-maximizing firm must be producing its optimal level of output at least cost.

(c) Solve this problem to derive the firm's supply curve. Use approximations where helpful.

Taking a derivative of the profit function with respect to q and setting it equal to zero yields

$$\frac{d\pi}{dq} = 0 \Longrightarrow p - \frac{4}{3}(3.78)q^{\frac{1}{3}} = 0 \Longrightarrow p \approx 5.04q^{\frac{1}{3}}$$

Cubing both sides yields $p^3 \approx 128q$, i.e., $q = \frac{p^3}{128}$. This is the firm's supply curve.

(d) If the price of output is $p = 16$, how much will the firm produce? What will its profits be?

Plugging $p = 16$ into the supply curve yields $q \approx 32$. So its profits are $\pi = pq - C(q) \approx 16(32) - 3.78(32)^{\frac{4}{3}} \approx 127.98$.

12. The previous problems have dealt with **long run** cost curves and supply curves, meaning that the firm has complete control over all of its inputs. In the **short run**, however, the firm cannot change its capital stock—it can choose how much labor to hire, but it can't build any new factories. In this problem we will examine short run cost curves and short run supply curves.

(a) Assume that the firm's production function is $f(L, K) = L^{\frac{1}{4}} K^{\frac{1}{2}}$, and that capital is fixed at $K = 4$. What is the equation for the isoquant corresponding to an output level of q?

The isoquant is $L^{\frac{1}{4}}(4)^{\frac{1}{2}} = q$, i.e., $L^{\frac{1}{4}} = \frac{q}{2}$, i.e., $L = \frac{q^4}{16}$. Note that the isoquant is not a line but *just a single point*. This is because capital is fixed at $K = 4$, so the firm has no ability to trade-off between capital and labor.

(b) Assume further that the prices of L and K are $p_L = 2$ and $p_K = 2$. Write down the problem for minimizing cost subject to the constraint that output must equal q. Clearly specify the objective function and the choice variables.

The firm wants to choose L to minimize $C(L, K) = 2L+2K = 2L+8$ subject to $f(L, K) = q$.

(c) In a previous problem you provided an intuitive explanation for the marginal rate of technical substitution. Given that capital is fixed at $K = 4$, what is the relevance (if any) of this concept in the present problem?

Since the amount of capital the firm has is fixed, the firm cannot substitute between labor and capital. So the marginal rate of technical substitution is irrelevant in this problem.

(d) How will the price of capital p_K affect the firm's behavior?

Capital is a sunk cost, so the price of capital will not affect the firm's behavior.

(e) Solve this problem, i.e., find the minimum cost $C(q)$ required to reach an output level of q. What is the optimal choice of L?

In order to produce output of q, the firm has to hire $L = \frac{q^4}{16}$ units of labor. So the cost of producing q units of output is $C(q) = 2(\frac{q^4}{16}) + 2(4) = \frac{1}{8}q^4 + 8$.

(f) Write down the profit maximization problem, using the function $C(q)$ you found above. Calculate the firm's short run supply curve.

The firm wants to choose q to maximize profits $\pi = pq - C(q) = pq - (\frac{1}{8}q^4 + 8)$. To solve this problem we take a derivative with respect to q and set it equal to zero, yielding

$$\frac{d\pi}{dq} = 0 \Longrightarrow p - \frac{1}{2}q^3 = 0 \Longrightarrow 2p = q^3 \Longrightarrow q = (2p)^{\frac{1}{3}}.$$

13. Consider a firm with production function $f(L, K) = L^{\frac{1}{2}}K^{\frac{1}{2}}$ and input prices of $p_L = 1$ and $p_K = 2$.

    (a) Calculate the supply curve for this firm. (Note: the related cost-minimization problem was done in the text, with an answer of $C(q) = 2q\sqrt{2}$.)

    The profit function is $\pi = pq - C(q) = pq - 2q\sqrt{2}$. Taking a derivative and setting it equal to zero we get $p - 2\sqrt{2} = 0$, i.e., $p = 2\sqrt{2}$.

    (b) How much will the firm supply at a price of $p = 2$? *Hint: Think about corner solutions!*

    At a price of $p = 2$, the firm will supply $q = 0$: its cost of producing each unit of output is $2\sqrt{2} > 2$, so it loses money on each unit it sells!

    (c) How much will the firm supply at a price of $p = 4$?

    At a price of $p = 4$, the firm will supply infinitely many units of output.

    (d) Show that this firm's production function exhibits **constant returns to scale**, i.e., that doubling inputs doubles output, i.e., $f(2L, 2K) = 2f(L, K)$.

    We have $f(2L, 2K) = (2L)^{\frac{1}{2}}(2K)^{\frac{1}{2}} = 2L^{\frac{1}{2}}K^{\frac{1}{2}} = 2f(K, L)$.

    (e) Does the idea of constant returns to scale help explain the firm's behavior when $p = 4$? *Hint: Think about this in the context of the objective function.* If it does help you explain the firm's behavior, you may find value in knowing that **increasing returns to scale** occurs when doubling inputs more than doubles outputs, i.e., $f(2L, 2K) > 2f(L, K)$, and that **decreasing returns to scale** occurs when doubling inputs less than doubles outputs, i.e., $f(2L, 2K) < 2f(L, K)$. An industry with constant or increasing returns to scale can often lead to monopolization of production by a single company.

    Since doubling inputs doubles output, the firm can double and redouble its profits simply by doubling and redoubling production (i.e., its choice of inputs). This (hopefully) helps explain why we get a corner solution (of $q = \infty$) when we attempt to maximize profits with $p = 4$.